THE
ASTROPHYSICAL JOURNAL
AN INTERNATIONAL REVIEW OF SPECTROSCOPY
AND ASTRONOMICAL PHYSICS

VOLUME XXXVI SEPTEMBER 1912 NUMBER 2

THE SYNTHETIC DEVELOPMENT OF RADIATION LAWS FOR METALS

By EDWARD P. HYDE

INTRODUCTION

A lively interest is being manifested at the present time in the radiating properties of metals. This has been stimulated, no doubt, by the utilization of the metals tantalum, tungsten, and osmium in the manufacture of high-efficiency lamps. An explanation of their high efficiencies has been sought, and this study has led to a consideration of the relative importance of the temperature of operation and the *selectivity*[1] of radiation in determining the efficiency. The primary source of interest to the physicist, however, lies in the development of the physics of radiation, which in turn conduces to a better knowledge of matter in various conditions of thermal equilibrium.

We seek, therefore, to correlate the radiating properties with the other known properties of material substances and the first step in such a correlation is the formulation of the law or laws which express the radiating properties. The laws of radiation for the *ideal radiator* or *black body* are fairly well known, though the exact values of the constants have not yet been determined with entire

[1] Throughout this paper the term *selectivity* will be used to indicate a difference in quality of radiation from that of a black body at the same true temperature. It will not signify the extreme variations from black-body radiation to which the term was originally applied, as, for example, in the *bright-line* spectra of luminous gases.

satisfaction. For ordinary matter, though much progress has been made in the study of the radiating properties of some few substances, our knowledge on the whole is very limited. Platinum and iron oxide with two or three possible exceptions are the only two substances for which any successful attempt[1] has been made to study directly the relation between temperature and radiation at temperatures above 1200° or 1500° abs.[2] The reason for this is to be found in the difficulty in making direct measurements of high temperature, particularly of those substances which cannot be operated in the open air. Consequently in the recent studies of the radiation of such substances as tantalum, tungsten, and carbon, recourse has been had to indirect methods. These indirect methods have yielded certain limited facts. Thus without any assumptions it has been shown[3] by experiment that carbon, tantalum, tungsten, and osmium radiate selectively with respect to a black body. On the basis of a single assumption, which in the light of all our present knowledge is quite probable, these experiments have made possible numerical estimates of the degree of selectivity of the radiation from these various substances, and have indicated certain definite temperature relationships. Other experiments[4] giving the distribution of energy in the infra-red region of the spectrum have led to the evaluation of the so-called *constant* a in the generalized form of the Wien equation

$$J=\frac{C_1}{\lambda^a e^{\frac{C_2}{\lambda T}}} \qquad (1)$$

which is supposed to represent fairly well, at least for short wavelengths and low temperatures, the distribution of energy in the spectrum of a black body when the constant $a=5$. These experiments have yielded values of a other than 5, and different for

[1] Lummer and Pringsheim, *Verh. d. deutsch. phys. Ges.*, **1**, 226, 1899; Paschen, *Wied. Annalen*, **58**, 455, 1896; *ibid.*, **60**, 662, 1897.

[2] Throughout this paper temperatures will be given in degrees *absolute*, i.e., degrees will be intervals on the Centigrade scale, but the zero will be the absolute zero, or —273° from the melting-point of ice.

[3] Hyde, Cady, and Middlekauff, *Trans. Ill. Eng. Soc.*, **4**, 334, 1909. See also *Ill. Eng.* (London), **2**, 241, 335, 1909.

[4] Coblentz, *Bulletin of the Bureau of Standards*, **5**, 361, 1909.

different substances. Moreover, they have indicated a variation in a with the temperature. The differences in a for different substances point to differences in the radiating properties of the various metals studied, but the variation in a with temperature would seem to suggest the inapplicability of the generalized Wien law to the radiation of these substances.

It has long been a matter of conjecture whether or not the radiation of certain metals can be represented by either the generalized Wien equation or the generalized Planck equation

$$J=\frac{C_1}{\lambda^a\left(e^{\frac{C_2}{\lambda T}}-1\right)} \tag{2}$$

which, when $a=5$, has a better foundation, both theoretical and experimental, as representing black-body radiation. The two equations are practically the same for short waves and low temperatures, but differ somewhat when the product λT becomes relatively large. Platinum, the only metal studied directly, seems to be represented fairly well in its radiation by the above equations, though it is doubtful whether the data on platinum justify discrimination between the Wien and Planck equations. By two different methods a has been determined for platinum as 6.0[1] and 6.42.[2] Moreover, the relation between the total emission and the temperature has been found to be approximately

$$E=\sigma T^{\beta} \tag{3}$$

where β lies between 5 and 5.5. If the generalized Wien law (equation 1) or the generalized Planck law (equation 2) holds for platinum, then it must follow by integration that the total emission is given by the generalized Stefan-Boltzmann law (equation 3) and that the exponent $\beta=a-1$.

It is thus seen that to a fair degree of approximation the radiation from platinum follows the radiation laws for a black body if the constants of these laws are changed. To what extent the radiation from platinum follows the black-body laws, is, however, not known exactly, and it is a distinct assumption to suppose that the

[1] Lummer and Pringsheim, *Verh. d. deutsch. phys. Ges.*, **1**, 218, 1899.

[2] Paschen, *Annalen der Physik*, **60**, 697, 1897.

radiation from other metals will follow the same laws by changing the constants in the black-body equations. For each metal this must be tested, and since, as stated before, it is most difficult to make direct temperature measurements, recourse must be had to indirect methods.

Two general methods of procedure are available. According to a method which has been used extensively the generalized black-body laws are assumed and then deductions from these laws are made and tested experimentally. If a number of these deductions are verified a high degree of probability is established that the assumption is correct. If, on the other hand, some deductions are verified and others are not it is very difficult to form any exact idea of the degree of approximation to which the assumed laws hold. Thus, if on the assumption of the generalized Wien and Planck equations, a for tungsten is found to be higher than that for tantalum, as determined by Paschen's method, some definite result would seem to have been obtained, but if a for each substance is further found to change with the temperature, or with some other variable, the inapplicability of the generalized Wien or Planck equation is at once evidenced, and doubt is cast on the conclusions from the first observation. This doubt can be dispelled only by a careful analysis of all the data. This method, which may be termed the *analytic*, is, however, a very valuable one, and in some cases is the only one available.

The other general method of procedure is the *synthetic* method. Starting from the observed phenomena, an attempt is made to build up the laws or generalizations which correlate and express the phenomena. After an unsuccessful effort to present in proper perspective by the analytical method a quantity of data which the author has accumulated on the radiation of carbon and of the metals tungsten and tantalum, the synthetic method suggested itself. It has proved itself a much more satisfactory way of correlating these and the other data available at the present time.

In the present paper an attempt will be made, as far as the available data will permit, to build up the equations which express the various relations between T, the absolute temperature, λ, the wave-length, J, the intensity of emission at any wave-length, and

E, the total emission for the various substances—carbon, tantalum, and tungsten. Observations made in the laboratory of the author will form the basis for the synthetic construction of the radiation laws, but the observations of others will also be utilized for confirmation and extension.

SYNTHETIC DEVELOPMENT OF RADIATION LAWS FOR CARBON, TANTALUM, AND TUNGSTEN

Criterion I

The synthetic development of the radiation laws for carbon, tantalum, and tungsten rests primarily on two experimental facts. One of these has already been published[1] and has been confirmed by subsequent experiments. It has been described as a relation between *color-match* and *intensity-match*, but will be referred to hereafter as Criterion I, since the term color-match has led to some confusion.

In order to state Criterion I, which is very closely fulfilled by two of the three substances mentioned above and approximately fulfilled by the third substance, let us assume a black body at some relatively low, known temperature T_0. Let $(J_1)_0$ and $(J_2)_0$ be the intensities of emission in any two wave-lengths λ_1 and λ_2 in the *visible* spectrum. Let us assume a second radiating body, e.g., tantalum, at such an unknown temperature T_1 that

$$\frac{(J_1)_1}{(J_2)_1}=\frac{(J_1)_0}{(J_2)_0} \tag{4}$$

where $(J_1)_1$ and $(J_2)_1$ are the intensities of emission in the wave-lengths λ_1 and λ_2 for tantalum at the temperature T_1.

Now suppose the temperature of the black body is increased to some higher, arbitrary known temperature T'_0, and that the temperature of the tantalum is increased by an unknown amount, such, however, that at the new temperature T'_1 the following relation holds:

$$\frac{(J'_1)_1}{(J'_2)_1}=\frac{(J'_1)_0}{(J'_2)_0} \tag{5}$$

where the primed quantities are the corresponding quantities at the higher temperatures. In previous papers the author has referred

[1] Hyde, *Ill. Eng.* (London), **3**, 488, 1910. See also *Physical Review*, **31**, 315, 1910.

to the conditions expressed by equations 4 and 5 as color-matches because, when λ_1 and λ_2 are two wave-lengths of the visible spectrum some distance apart, it has been found, by repeated experiment for the three substances under consideration, that when equations 4 and 5 are fulfilled as determined by spectrophotometric measurements, the integral colors of the two radiating bodies are indistinguishable as seen in the ordinary Lummer-Brodhun contrast photometer. In fact the latter criterion is probably more sensitive than the spectrophotometric determination. In order, however, to avoid any possible criticism, the discussion, at least for the present, will be based on the spectrophotometric observations as given in equations 4 and 5.

Suppose now that when a black body and tantalum are at such temperatures T_0, T'_0, T_1, and T'_1 that equations 4 and 5 are fulfilled, there is also fulfilled this condition:

$$\frac{(J'_1)_1}{(J_1)_1}=\frac{(J'_1)_0}{(J_1)_0}. \tag{6}$$

Then Criterion I is fulfilled and certain interesting conclusions can be drawn. Previous experiments[1] confirmed by more recent ones have shown that Criterion I is fulfilled within the limits of accuracy of the experiments for carbon and tantalum and that if there is a deviation in the case of tungsten this discrepancy is small. The experimental data will be presented later. (The approximate method of integral color-match gives one a better idea of the significance of the criterion. If two lamps, one of untreated carbon and one of tantalum, are mounted on opposite sides of a sensitive Lummer-Brodhun photometer and are brought to a color-match first at low voltages, and then at higher voltages, it will be observed that the relative intensities of the two lamps remain the same notwithstanding the different radiating properties of the two substances.)

If Criterion I as expressed by the three equations 4, 5, and 6 is fulfilled for all wave-lengths of a given region of the spectrum, it can readily be shown that for this region of the spectrum

$$\frac{d(J)_1}{dT_1}\cdot\frac{T_1}{(J)_1}=\frac{A}{\lambda} \tag{7}$$

[1] *Loc. cit.*

in which A is independent of λ but may be a function of the temperature. Equations 4, 5, and 6 give directly

$$\frac{(J'_2)_1}{(J_2)_1}=\frac{(J'_2)_0}{(J_2)_0}. \tag{8}$$

Since the criterion is supposed to hold for all temperatures over a relatively large range, let us take the two temperatures T_0 and T'_0 near together, such that $T'_0=T_0+\Delta T_0$ where Δ is a small fraction. Equations 6 and 8 then become

$$\left(\frac{\Delta J_1}{J_1}\right)_1=\left(\frac{\Delta J_1}{J_1}\right)_0 \tag{9}$$

and

$$\left(\frac{\Delta J_2}{J_2}\right)_1=\left(\frac{\Delta J_2}{J_2}\right)_0. \tag{10}$$

Dividing equation 10 by equation 9 there results

$$\left(\frac{\dfrac{\Delta J_2}{J_2}}{\dfrac{\Delta J_1}{J_1}}\right)_1=\left(\frac{\dfrac{\Delta J_2}{J_2}}{\dfrac{\Delta J_1}{J_1}}\right)_0. \tag{11}$$

If now ΔT_0 is made to approach zero, it must follow, since the changes in J_2 and in J_1 within each parenthesis, i.e., for each substance, correspond to the same changes in temperature, ΔT_0 for the black body and the corresponding ΔT_1 for tantalum, that

$$\left[\left(\frac{\dfrac{dJ}{J}}{\dfrac{dT}{T}}\right)_{\lambda_2}\div\left(\frac{\dfrac{dJ}{J}}{\dfrac{dT}{T}}\right)_{\lambda_1}\right]_1=\left[\left(\frac{\dfrac{dJ}{J}}{\dfrac{dT}{T}}\right)_{\lambda_2}\div\left(\frac{\dfrac{dJ}{J}}{\dfrac{dT}{T}}\right)_{\lambda_1}\right]_0. \tag{12}$$

The right-hand member, since it refers to a black body, can readily be shown to equal $\dfrac{\lambda_1}{\lambda_2}$. Thus, taking the derivative of the Wien equation (which is identical with the Planck equation in the visible spectrum for the temperature region in which we are interested at present) there results

$$\frac{dJ}{dT}=\frac{d}{dT}\;\frac{C_1}{\lambda^5 e^{\frac{C_2}{\lambda T}}}=\frac{C_1}{\lambda^5 e^{\frac{C_2}{\lambda T}}}\;\frac{C_2}{\lambda T^2}$$

so that

$$\frac{dJ}{dT}\cdot\frac{T}{J}=\frac{\dfrac{dJ}{J}}{\dfrac{dT}{T}}=\frac{C_2}{\lambda T}. \tag{13}$$

Hence

$$\left[\left(\frac{\dfrac{dJ}{J}}{\dfrac{dT}{T}}\right)_{\lambda_2}\div\left(\frac{\dfrac{dJ}{J}}{\dfrac{dT}{T}}\right)_{\lambda_1}\right]_0=\frac{\lambda_1}{\lambda_2} \tag{14}$$

and so (equation 12)

$$\left[\left(\frac{\dfrac{dJ}{J}}{\dfrac{dT}{T}}\right)_{\lambda_2}\div\left(\frac{\dfrac{dJ}{J}}{\dfrac{dT}{T}}\right)_{\lambda_1}\right]_1=\frac{\lambda_1}{\lambda_2}. \tag{15}$$

It must follow, therefore, that for tantalum

$$\frac{d(J)_1}{dT_1}\,\frac{T_1}{(J)_1}=\frac{A}{\lambda} \tag{16}$$

as given in equation 7, A is independent of λ but may be a function of T. Equation 16 may therefore be written, if at the same time the subscripts are dropped

$$\frac{dJ}{dT}\,\frac{T}{J}=\frac{A(T)}{\lambda}. \tag{17}$$

This equation must hold for any substance, and for any ranges of temperature and wave-length for which Criterion I is fulfilled. The limits of wave-length and temperature within which it applies for carbon, tantalum, and tungsten will be discussed later.

Now by integration of equation 17 a relation between J, λ, and T is obtained, corresponding to the Wien or Planck equation for a black body. Thus

$$\frac{dJ}{J}=\frac{1}{\lambda}A(T)\frac{dT}{T}$$

$$\log J=\frac{1}{\lambda}\int A(T)\frac{dT}{T}+B(\lambda)$$

$$J=e^{B(\lambda)}\cdot e^{\frac{1}{\lambda}\int A(T)\frac{dT}{T}}. \tag{18}$$

If we write

$$\left.\begin{aligned} e^{B(\lambda)} &= C_1 f(\lambda) \\ \int A(T)\frac{dT}{T} &= -\frac{C_2}{\theta(T)} \end{aligned}\right\} \quad (19)$$

equation 18 becomes

$$J = C_1 f(\lambda) \frac{1}{e^{\frac{C_2}{\lambda\theta(T)}}} \quad (20)$$

which is the generalized equation of energy distribution in the spectrum of the substance fulfilling Criterion I within the prescribed limits of wave-length and temperature. If $f(\lambda)=\lambda^{-5}$ and $\theta(T)=T$, equation 20 reduces to the Wien equation. If $f(\lambda)=\lambda^{-a}$ and $\theta(T)=T$, equation 20 reduces to the generalized form of the Wien equation, which has been found to hold approximately for platinum and which has been assumed in the so-called analytical method of studying other metals. It is seen, however, that Criterion I by itself does not demand that $f(\lambda)$ should be a power function. Moreover, it gives no information regarding the nature of the function $\theta(T)$.

It is interesting to consider in connection with Criterion I certain experimental facts established by other observers. It has been shown in various investigations[1] that in the region of the visible spectrum the optical constants of certain metals that have been studied undergo no change with change in temperature. According to Hagen and Rubens[2] the emissive power of metals is independent of temperature, not only in the visible spectrum, but even in the infra-red region up to and beyond 2 μ, notwithstanding the equation derived from the electromagnetic theory of Maxwell according to which the emissivity is proportional to $\sqrt{\frac{\rho}{\lambda}}$ where ρ is the specific resistance of a metal, and hence is, in general, a function of the temperature. For the region of the spectrum and the range of temperature within which the emissivity of a metal

[1] Pflüger, *Wied. Annalen*, **58**, 494, 1896.

[2] Hagen and Rubens, *Phys. Zeit.*, **11**, 139, 1910.

does not change with the temperature the equation representing the distribution of energy may at once be written as follows:

$$J = C_1 F(\lambda) \cdot \frac{1}{\lambda^5 e^{\frac{C_2}{\lambda T}}} \tag{21}$$

since the distribution of energy is obtainable from that of a black body at the same temperature by multiplying by some function of the wave-length $F(\lambda)$ which is independent of the temperature. Since in equation 21 the specific function $\frac{C_2}{T}$ takes the place of the more general function $\frac{C_2}{\theta(T)}$ in equation 20, being the only difference between the two equations, it is seen that the experimental fact of the constancy of the emissivity of a metal for changes in temperature conduces to more definite knowledge of its law of radiation (within the prescribed limits of wave-length and temperature) than Criterion I. The reason for this, and the physical significance of Criterion I are evident on a little consideration.

Differentiation of equation 21 leads to the expression

$$\frac{dJ}{dT} \cdot \frac{T}{J} = \frac{C_2}{\lambda T} \tag{22}$$

which does not involve the function $F(\lambda)$. According to equation 22 the percentage increase in intensity of emission $\frac{dJ}{J}$ in wave-length λ corresponding to a given small percentage increase in temperature $\frac{dT}{T}$ will be inversely as the wave-length λ. Hence if two substances, radiating in accordance with equation 21 with different functions $F(\lambda)$ and $F'(\lambda)$, are at such temperatures that $\left(\frac{J_{\lambda_1}}{J_{\lambda_2}}\right)_1$ for one substance is the same as $\left(\frac{J_{\lambda_1}}{J_{\lambda_2}}\right)_2$ for the other, and if the temperature of one is increased by a given small amount and the temperature of the other is increased by such an amount that again the condition is fulfilled that $\left(\frac{J'_{\lambda}}{J'_{\lambda_2}}\right)_1 = \left(\frac{J'_{\lambda_1}}{J'_{\lambda_2}}\right)_2$, it must follow that the relative change in intensity of emission in either wave-length λ_1 or λ_2 is the same for the two substances. In other words,

Criterion I is fulfilled. Indeed, since equation 21 is only a specialized form of equation 20, it follows at once that for two bodies which radiate according to equation 21, that is, which have emissivities independent of the temperature, Criterion I must be fulfilled.

The converse of this is not necessarily true, viz., that if Criterion I is fulfilled by a radiating substance in comparison with a black body then the emissivity of that substance is independent of the temperature within the limits of wave-length and temperature within which Criterion I holds. If Criterion I is fulfilled one of two conditions must exist: either the emissive power of the substance has remained constant, or else a change in the relative emissive power in the various wave-lengths of the visible spectrum has accompanied a change in the average emissivity in this region. Thus if the average emissive power in the visible spectrum increases, the increase for the shorter wave-lengths must be greater by a definite amount than that for the longer wave-lengths of the visible spectrum—otherwise Criterion I would not be fulfilled.

According to the results of Hagen and Rubens, however, the change would most probably be in the other direction. Thus for metals having a positive temperature coefficient of resistance the increase in emissivity with increasing temperature is found to be greater for the longer infra-red waves than for the shorter ones. If this change in emissivity progresses into the region of the visible spectrum at very high temperatures it would seem most probable that the emissivity in the red would increase more rapidly than that in the blue, rather than that the effect should be in the opposite direction. But if the phenomenon proceeds in the way which seems more probable, the relative change in emissivity in the visible would be in such a direction as to augment the effect of the average change, rather than to compensate it, so that the fulfilment of Criterion I points strongly to the constancy of the emissivity of the substance within the prescribed limits of wave-length and temperature, rather than to the alternate conclusion.

Criterion I lends itself admirably to the study of the constancy of the emissivity of metals at high temperatures in that region of the infra-red spectrum where Hagen and Rubens found a transition from the complete fulfilment of the equation resulting from the

electromagnetic theory, to the condition of constant emissivity as found in the visible spectrum.

Criterion II

The second experimental fact, which forms the basis of Criterion II, involves the total radiation from the substance under investigation. The Stefan-Boltzmann law of total radiation for a black body

$$E=\sigma_0 T^4 \tag{23}$$

has both theoretical and experimental foundation. Moreover, if either the Wien or the Planck law of spectral energy distribution is true for the black body the Stefan-Boltzmann law must follow from integration, and the constant σ_0 must be expressible in terms of the constants entering into the Wien or Planck equation. Thus from the generalized Planck equation

$$J=\frac{C_1}{\lambda^a(e^{\frac{C_2}{\lambda T}}-1)} \tag{24}$$

it must follow that

$$E=\int_0^\infty J d\lambda=\int_0^\infty \frac{C_1}{\lambda^a(e^{\frac{C_2}{\lambda T}}-1)}$$

or

$$E=\left[C_1\Gamma(a-1)\frac{1}{C_2^{a-1}}\sum_{n=1}^{n=\infty}\frac{1}{n^{a-1}}\right]T^{a-1}. \tag{25}$$

The constant σ in the generalized Stefan-Boltzmann law

$$E=\sigma T^{a-1} \tag{26}$$

must then be (equation 25)

$$\sigma=C_1\Gamma(a-1)\frac{1}{C_2^{a-1}}\sum_{n=1}^{n=\infty}\frac{1}{n^{a-1}}. \tag{27}$$

If the Wien equation is used instead of the Planck equation, this becomes

$$\sigma'=C_1\Gamma(a-1)\frac{1}{C_2^{a-1}}. \tag{28}$$

If a is an integer, as in the case of a black body, $\Gamma(a-1)=(a-2)!$ and so for a black body, for which $a=5$, equations 27 and 28 reduce to

$$\sigma_0=C_1(3!)\frac{1}{C_2{}^4}\sum\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots\ldots, \text{ etc.} \tag{29}$$

and

$$\sigma'_0=\frac{6C_1}{C_2{}^4}. \tag{30}$$

This simple process of arriving at the law of total radiation by integration of the Wien or Planck equation of spectral energy distribution is not applicable to equations 20 or 21 expressing the distribution of energy in the spectrum of certain substances, because these equations have been formed only from data in the visible spectrum. Other experimental data involving the total radiation are therefore necessary. These data are supplied in the second experimental fact forming the basis of Criterion II.

This second experimental fact is based on accurate photometric and electrical measurements on various incandescent electric lamps[1] from which is derived the relation between the supplied watts W and the coefficient k', where k' is defined as the ratio of the relative change in candle-power, I, to the relative change in the supplied watts, W, or

$$k'=\frac{\frac{dI}{I}}{\frac{dW}{W}}=\frac{dI}{dW}\cdot\frac{W}{I}. \tag{31}$$

From the observed values of k' and W it is possible to pass to the relation between the radiated energy E and the coefficient k_λ, where

$$k_\lambda=\frac{\frac{dJ}{J}}{\frac{dE}{E}}=\frac{dJ}{dE}\cdot\frac{E}{J}. \tag{32}$$

It is thus the ratio of the relative change in the emission, J, at some wave-length, λ, to the total emission, E. A previous investi-

[1] Cady, *Trans. Ill. Eng. Soc.*, **3**, 459, 1908; *Elec. Rev. and West. Elec.*, **59**, 1087, 1911; *Ztschr. f. Beleuch.*, **18**, 109, 1912.

gation by the author[1] and others showed how the total radiation, E, for a unit length of a filament can be determined from the total supplied watts, W, and how the candle-power for a unit length of filament can be determined from the average candle-power of the entire filament. The simple method of passing accurately from the relative change in this integral or white-light candle-power to the relative change in the intensity, J, at some wave-length in the visible spectrum will be given subsequently in presenting the experimental data.

The second experimental fact, which has been observed to hold for carbon, tantalum, and tungsten, is the following relation between k_λ (equation 32) and E:

$$\frac{(k_\lambda)_1}{(k_\lambda)_2}=\left(\frac{E_2}{E_1}\right)^{1/\beta} \tag{33}$$

where $(k_\lambda)_1$ and $(k_\lambda)_2$, and E_1 and E_2 are values of k_λ and E, respectively, corresponding to the temperatures T_1 and T_2. β is a constant having different values for the different substances. The constancy of β with change of temperature forms the basis of Criterion II.

To determine the significance of Criterion II let equation 33 be rewritten in the following equivalent form

$$k_\lambda=\frac{K}{E^{1/\beta}} \tag{34}$$

in which K is a constant factor of proportionality. Now k_λ from its definition (equation 32) may be written

$$k_\lambda=\frac{dJ}{dE}\cdot\frac{E}{J}=\frac{dJ}{dT}\cdot\frac{T}{J}\div\frac{dE}{dT}\cdot\frac{T}{E} \tag{35}$$

and the differential equation from which equation 20 was deduced is (equation 17)

$$\frac{dJ}{dT}\cdot\frac{T}{J}=\frac{A(T)}{\lambda}. \tag{36}$$

If we assume that Criterion I indicates the constancy of the emissivity with change in temperature, as discussed fully above,

[1] Hyde, Cady, and Worthing, *Trans. Ill. Eng. Soc.*, **6**, 238, 1911. See also *Ill. Eng.* (London), **4**, 389, 1911.

then the function $A(T)$ will be replaced by $\frac{C_2}{T}$ (by differentiation of equation 21) and equation 36 becomes

$$\frac{dJ}{dT} \cdot \frac{T}{J} = \frac{C_2}{\lambda T} \cdot \tag{36'}$$

These two equations (36 and 36′), based on equations 20 and 21 respectively, will be carried through in parallel for completeness.

Substituting in equation 35 the values of $\frac{dJ}{dT} \cdot \frac{T}{J}$ as given in equations 36 and 36′ and the value of k_λ as given in equation 34, there result respectively

$$\frac{dE}{E^{(1/\beta)+1}} = \frac{1}{\lambda K} \cdot A(T) \frac{dT}{T} \tag{37}$$

and

$$\frac{dE}{E^{(1/\beta)+1}} = \frac{C_2}{\lambda K} \frac{dT}{T^2} \cdot \tag{37'}$$

On integration these become

$$E = S'[\theta(T)]^\beta \tag{38}$$

and

$$E = ST^\beta \, . \tag{39}$$

The first of these two equations (equation 38) follows rigorously from Criteria I and II; the other (equation 39) follows in the special case that the emissivity is independent of the temperature. In the latter case it is seen, from a comparison of equations 34 and 39, that

$$Tk_\lambda = \text{constant} \tag{40}$$

so that T, the absolute temperature, is inversely proportional to the observed quantity k_λ.

EXPERIMENTAL DATA

Criterion I

As stated in previous paragraphs, Criterion I furnishes a means of determining the probable constancy of the emissivity of radiating substances in the visible spectrum over any range of temperature within which the criterion is applicable. The criterion was first used[1] as a means of testing the variation of a with temperature

[1] *Loc. cit.*

in the generalized Wien equation (equation 1) on the assumption of which Coblentz[1] had found a to decrease generally with temperature. Coblentz had determined a by Paschen's method from infra-red energy measurements. The author, by the application of Criterion I, found that there was little indication of a variation of a with temperature in the visible spectrum. In these early experiments ordinary commercial lamps were used, and no effort was made to avoid the effects produced by the cooling of the filaments at the leading-in and supporting wires.

The fulfilment of Criterion I, however, has more significance than was originally appreciated, and it was in the light of this new significance that the recent experiments were conducted. If Criterion I is fulfilled in the comparison of a metal with a black body, it points strongly to the constancy of the emissivity of the metal within the range of temperature and wave-length employed. It should be emphasized, however, as already stated, that Criterion I would also be fulfilled even though the average emissivity in the visible spectrum underwent some change with temperature, provided that this change in the average emissivity was accompanied by a relative change in the emissivity at different wave-lengths, of the right magnitude, and in the proper direction. This alternative significance is well within the range of possibility, but it would seem much more probable that the fulfilment of Criterion I would arise from the constancy of the emissivity in the visible spectrum.

In the recent experiments intercomparisons were made between untreated carbon, treated carbon, tantalum, tungsten, and the electrically heated black body. Various lamps were used, some of them of the ordinary commercial type, and others of special construction so made as to reduce the effect of the end losses to a minimum. In the case of the special lamps the most accurate set of measurements was made using a diaphragm to screen off the cooled ends of the filament. The exact procedure will be outlined below. Moreover, measurements were made in the two ways: (1) using the integral light from the lamps and bringing the two lamps to a color-match by means of an ordinary Lummer-Brodhun contrast photometer; (2) using monochromatic radiation of two wave-

[1] *Loc. cit.*

lengths and bringing the two lamps to the same relative intensities in these chosen wave-lengths by the use of a Lummer-Brodhun spectrophotometer. In all the experiments the same general results were obtained, but the conclusion reached several years ago in the first experiments,[1] using a color-match method, was corroborated, viz., that two lamps (having energy curves of the black-body type) can be brought to the same relative intensity of energy emission in two wave-lengths of the visible spectrum more accurately by means of the color-match method using the integral light than by means of direct measurements in the two wave-lengths as with a spectrophotometer.

The most trustworthy measurements are those which were made using this method and employing specially constructed lamps with long free filaments, the cooled ends of the filaments being screened by diaphragms placed between the lamp and the photometer, and as close to the lamp bulb as possible. The procedure was as follows: An untreated carbon filament lamp was mounted on one side of the photometer and a flashed or treated carbon filament lamp on the other side, each being so screened that only light from that portion of the filament which was of uniform brightness reached the photometer. The untreated carbon lamp was set at some arbitrary low voltage and then the voltage of the flashed carbon lamp was varied until a color-match was obtained as seen in the Lummer-Brodhun contrast photometer. Several determinations of the voltage of the flashed carbon lamp were made and the average value taken. The untreated carbon lamp was then set at some arbitrary high voltage, the range in voltage being made as large as possible consistent with the accuracy of color measurement at the lower limit, and the constancy of the lamp at the upper limit. A second set of measurements was made at this high voltage to determine the voltage of color-match for the flashed carbon lamp, and the average value taken.

Having determined the voltages of the flashed carbon lamp corresponding to color-match with the untreated carbon lamp at the two arbitrary voltages, the two lamps were intercompared as to relative candle-power when they were operated at the color-

[1] *Loc. cit.*

match voltages at the lower temperatures, and again when they were operated at the color-match voltages at the higher temperatures. If Criterion I were fulfilled the relative candle-powers should be the same in the two cases. If any difference in the relative candle-powers had been found Criterion I would have been shown not to be fulfilled.

A set of measurements such as that outlined above was made by at least two different observers for the four radiating substances —untreated carbon, flashed carbon, tantalum, and tungsten. In each case the untreated carbon lamp was placed on one side of the photometer and operated at the same two voltages, the other substances being placed successively on the other side of the photometer. The results of these measurements are given in Table I. In the first column are given the lamps which were compared with the untreated carbon lamp No. 63 operated at the two voltages $V_0 = 35$ volts and $V'_0 = 50$ volts. In the third column are given the individual determinations of $\frac{I'}{I'_0} \div \frac{I}{I_0}$ where I_0 and I'_0 are the candle-powers of the untreated carbon lamp at the two voltages V_0 and V'_0, and I and I' are the respective candle-powers of the other substances at the voltages V and V' corresponding to color-matches with the untreated carbon lamp at its two voltages. In the fourth column are given the average values. A quantity larger than unity indicates that the emissivity of the substance increases more rapidly than that of the untreated carbon and vice versa.

Taking untreated carbon as the standard, it is seen that there is no indication of a relative change in the emissivity in the visible spectrum of treated carbon or tantalum, but that there is marked evidence of a change in the emissivity of tungsten compared with that of the other substances. It is true that the data presented would indicate an increase in emissivity of 0.2 per cent for treated carbon, and a decrease of 0.7 per cent for tantalum, compared with untreated carbon, but it should be emphasized that the accuracy attainable in these measurements is certainly not better than 1 per cent and is more likely of the order of magnitude of 2 or 3 per cent. Indeed, the change of 3 or 4 per cent observed in the case of tungsten is so nearly within the reasonable limits of accuracy attainable

that it was at first thought to have no definite significance, but as every one of the number of distinct experiments performed at different times and under different conditions, including one or two made with the spectrophotometer using monochromatic light of two wave-lengths, gave a deviation in the same direction and of the same order of magnitude, the reality of the indicated change seemed to be established.

The degree of uncertainty in the measurements is determined by the accuracy with which the color-match can be made, in the experiments with white light, or with which the filaments can be brought to the same relative intensity of emission in the two chosen wave-lengths, in the experiments with monochromatic light. As has already been stated, experience indicates that the two methods yield the same result within the limits of accuracy that have been reached, and that the color-match method is quite as sensitive, perhaps even a little more sensitive than the spectrophotometric method using monochromatic light, which is theoretically preferable, but which is very tedious. For these reasons the principal set of experiments of which the results are given in Table I was carried out using the color-match method. In a later part of the paper (Fig. 3) spectrophotometric curves corresponding to color-match will be given.

So far nothing has been said regarding absolute measurements of emissivity by comparison with a black body. The experiments cited indicate that untreated carbon, flashed carbon, and tantalum have no relative change in emissivity in the visible spectrum within the limits of temperature employed (about 400° centering around 2000° or 2100° abs.), and that the emissivity of tungsten decreases slightly in going from low to high temperature. Without comparison with a black body, however, it is impossible to say with certainty that the emissivities of untreated carbon, flashed carbon, and tantalum in the visible spectrum are constant, and that the emissivity of tungsten decreases slightly, although this conclusion may appear most probable. It may be that the emissivity of tungsten is constant and that the emissivities of the other filament materials increase slightly with increasing temperature, or even that the emissivities of all four bodies are changing, but that the differential changes are as indicated.

With the idea of settling the above question, Criterion I was applied in a comparison between an electrically heated black body and untreated carbon, using the color-match method. This test was not entirely satisfactory owing to the difficulty of securing a uniformly heated cavity with the furnace at the author's disposal at this time. The result obtained in the experiment indicated that the emissivity of the untreated carbon remained quite constant, but under the conditions an accuracy better than 2 or 3 per cent could not be considered. Hence so far as this experiment is concerned the only conclusion is that the emissivities of the entire group of bodies under investigation remain sensibly constant—the observed changes given in Table I being of the order of magnitude of the absolute changes, and not merely changes of a second order.

This conclusion is further strengthened by computations from earlier experiments made, with a different purpose in mind, on two electrically heated black bodies belonging respectively to Drs. Waidner and Burgess, and to Dr. Coblentz at the Bureau of Standards. Color-match measurements over a range of temperature were made between the two black bodies and a tungsten lamp at various voltages. As the tungsten lamp is still available it has been possible to measure its candle-power and to compare this with the computed change in candle-power of the black body over a given temperature range. The results of this comparison strengthened the conclusion from the other experiment that no large changes in emissivity in the visible spectrum occur in the cases of the filament materials studied.

In the light of all the experiments it would seem probable, therefore, that the emissivities of untreated carbon, flashed carbon, tantalum, and tungsten in the visible spectrum undergo little change over a temperature range of 300° or 400° centering around 1900°–2100°; further, that the observed relative change between tungsten and the other bodies is more probably to be ascribed to a slight change in the emissivity of tungsten, the emissivities of the other substances remaining constant. That the difference between tungsten and the other substances is real was established by a comparison of tungsten directly with tantalum over a much more extended range of temperature than was possible with untreated

carbon. This test gave as a mean of four sets of measurements, two sets by each of two observers, a difference in the average of $6\frac{1}{2}$ per cent, the direction being the same as before, viz., that the emissivity of tungsten decreases in going from low to high temperature, as compared with tantalum.

The constancy of the emissivity of metals with change in temperature has been shown by various other observers using entirely different methods. Sissingh,[1] Drude,[2] Pflüger[3] found little if any change in the optical constants of iron, silver, gold, platinum, etc., over a temperature interval of 100° or 200° C. near room temperature. Zeeman[4] studied the reflecting power of platinum up to 800° and failed to find any appreciable change over this range of temperature. Holborn and Henning[5] obtained similar results for gold, silver, and platinum. Finally Henning,[6] using his spectral pyrometer, found no indication of change in emissivity in the visible spectrum over large ranges of temperatures in the neighborhood of 1500°–2000° abs. and for a large variety of substances including carbon, tantalum, tungsten, osmium, platinum, etc. It is true that Henning's test, like that of the author, does not necessarily indicate constancy in emissivity. If the logarithm of the absorptivity (A_0) at $\lambda_0 = 0.5\,\mu$ increased $\frac{\lambda_1}{\lambda_0}$ times as fast as $\log A_1$, where A_1 is the absorptivity at $\lambda_1 = 0.7\,\mu$, then (see Henning's equations)

$$\theta_1 = \frac{1}{C}[\lambda_0 \log A_0 - \lambda_1 \log A_1]$$

would be constant even though A_0 and A_1 should vary. The observed constancy of θ, however, would seem in all probability to point strongly to the constancy of the absorptivity at both wave-lengths. Certainly all the data, taken together, indicate

[1] Sissingh, "Mesures de la polarisation elliptique de la lumiere," *Arch. Neerlandaises*, **20**. See also Dissertation, Leiden, 133, 1885.

[2] Drude, *Wied. Annalen*, **39**, 538, 1890.

[3] Pflüger, *ibid.*, **58**, 494, 1896.

[4] Zeeman, *Communications from the Laboratory of Physics at the University of Leiden*, No. 20, 1895.

[5] Holborn and Henning, *Berl. Berichte*, 311, 1905.

[6] Henning, *Ztschr. f. Instr.*, **30**, 61, 1910.

strongly that the emissivities of metals in the visible spectrum do not undergo any appreciable change with change in temperature.

The emissivities of certain metals in the infra-red regions of the spectrum have, on the contrary, been found to undergo marked change with the temperature. Hagen and Rubens have found that the emissivities in the infra-red regions beyond 6 μ conform in their changes with temperature to the following equation, derived from the electromagnetic theory:

$$\text{Emissivity} = 36.5\sqrt{\frac{\rho}{\lambda}}$$

where ρ is the specific resistance and λ is the wave-length. This equation is derived on the assumption of free electrons and under the condition that neither the electrical conductivity nor the period of the emitted light is large. Hence it is not surprising that Hagen and Rubens found the equation to hold for long waves, but not for short ones. Their experiments on platinum gave no indication of a change in emissivity with temperature at $\lambda = 2\ \mu$ over a temperature interval extending from 400° to 1400°. At $\lambda = 4\ \mu$ a marked increase in emissivity with increasing temperature was observed, though the magnitude of the change was not as large as that demanded by the equation given above. At $\lambda = 6\ \mu$ the equation was quite accurately verified. Observations on other substances gave similar results.

The results which the author found on the change in emissivity in the visible spectrum of tungsten as compared with those of carbon and tantalum stand out uniquely, and merit further investigation, for although Criterion I might be fulfilled and yet the emissivity might undergo change, it is impossible to imagine that Criterion I should fail to be fulfilled and still that there should be no change with temperature of the emissivity of one of the two substances compared.

In the fifth column of Table I are given the ratios of the square roots of the resistances of the various filaments at high and low temperatures as compared with the corresponding ratios for untreated carbon. If the equation cited above were fulfilled for carbon, tantalum, and tungsten for emissivity in the visible spec-

trum, then the values given in the fourth column for $\frac{I'}{I'_0} \div \frac{I}{I_0}$ should be equal to the corresponding values of $\sqrt{\frac{R'}{R'_0} \div \frac{R}{R_0}}$ given in the fifth column. That is, the emissivity of tungsten should have been found to increase about 20 per cent compared with untreated carbon, rather than decrease $3\frac{1}{2}$ per cent as observed.

TABLE I

CHANGES IN CANDLE-POWER (I) AND RESISTANCE (R) OF VARIOUS RADIATING BODIES, AS COMPARED WITH CANDLE-POWER (I_0) AND RESISTANCE (R_0) OF UNTREATED CARBON, CORRESPONDING TO COLOR-MATCH WITH UNTREATED CARBON AT $V_0 = 35$ VOLTS AND $V'_0 = 50$ VOLTS

Lamps	Observers	$\frac{I'}{I'_0} \div \frac{I}{I_0}$ Individual Determinations	Means	$\sqrt{\frac{R'}{R'_0} \div \frac{R}{R_0}}$
Untreated carbon No. 63 (taken as standard)	.....	.....	1.000	1.000
Flashed carbon No. 40	F.E.C.	0.999	1.002	
Flashed carbon No. 40	A.G.W.	1.006		1.04
Tantalum No. 6	F.E.C.	1.004		
Tantalum No. 6	E.P.H.	0.997	0.993	1.05
Tantalum No. 15	F.E.C.	0.979		
Tungsten No. 79	A.G.W.	0.965		
Tungsten No. 79	F.E.C.	0.967		
Tungsten No. 79	E.P.H.	0.954	0.963	1.19
Tungsten No. 82	A.G.W.	0.959		
Tungsten No. 82	F.E.C.	0.971		
Black body	.....	.....	1.00	

Attention should be called, in passing, to the applicability of Criterion I (using two wave-lengths) to the study of the change in emissivity with temperature in the infra-red spectrum by spectrobolometric measurements. The criterion does not require a knowledge of the temperature, and, moreover, is quite sensitive. It is hoped that such application of the method to various metals will be made in this laboratory.

EQUATION OF ENERGY DISTRIBUTION

In the light of the results of the application of Criterion I, taken together with the evidence of other investigations, it would seem

fairly well established that the emissivities in the visible spectrum of untreated carbon, flashed carbon, tantalum, and tungsten undergo little change with change in temperature over a temperature interval of about 400° centering around 2000° or 2100° abs. The observed difference between tungsten and the other three filament materials, which appears to be larger than the experimental error, will be ignored in the following discussion and the conclusions for each material will be correct only to the extent to which the constancy of the emissivity is verified.

Assuming, then, the constancy of the emissivity, the radiation from each of the five materials is distributed throughout the visible spectrum in accordance with equation 21 (with the modification according to Planck)

$$J = C_1 F(\lambda) \frac{1}{\lambda^5 (e^{\frac{C_2}{\lambda T}} - 1)} \tag{41}$$

in which C_2 is the black-body constant, but $C_1F(\lambda)$ depends on the material and is different for each substance. Two questions arise: (1) Over what range of wave-lengths does the equation hold? and (2) What is the form of the function $F(\lambda)$? In reply to the first question, the equation holds over the range of wave-lengths for which the emissivity is independent of the temperature. At 2 μ Hagen and Rubens found the emissivity constant for platinum and for an alloy of platinum-rhodium. At 4 μ the emissivity changes somewhat with the temperature. At what wave-length the emissivity begins to change is not known, but if it remains constant to 3 μ then the emissivity is constant throughout by far the largest part of the spectrum for platinum at the temperature of 1673° abs., the highest temperature (1400° C.+273°) at which the measurements on platinum were made. If the constancy of the emissivities of the substances with which we are dealing is established to 3 μ at temperatures of about 2000°, then the total energy will vary with the temperature quite approximately according to the integral of equation 41 taken over the entire spectrum. This is on the assumption that the emissivity in the ultra-violet undergoes no marked change with temperature, a condition which probably holds, particularly if there are no marked absorption bands. It has not

yet been established, however, for the substances under investigation that the emissivities remain constant out to 2.5 μ or 3 μ.

The answer to the second question is found in the absorption coefficients as computed from Coblentz'[1] measurements of the reflecting power of the various materials in the visible spectrum. These absorptivities (or emissivities) show clearly the selectivity in favor of the visible spectrum for graphite, tantalum, and tungsten at room temperature. And since the constancy of the emissivity in the visible spectrum is assumed, they hold for glowing temperatures also.

It is of interest to know whether $F(\lambda)$ can be represented by a power of λ, since in that case the energy distribution, over the range of wave-length for which the emissivity is constant, would be represented by the generalized form of the Planck equation (equation 2)

$$J = C_1 \frac{1}{\lambda^a (e^{\frac{C_2}{\lambda T}} - 1)}$$

where $F(\lambda)$ is replaced by $\frac{1}{\lambda^{a-5}}$. It was concluded, however, from computations using Coblentz' data that no power $a-5$ could be obtained so that $\frac{1}{\lambda^{a-5}}$ would represent $F(\lambda)$. Fig. 1 shows the relative distributions of energy in the visible spectrum according to the generalized Planck equation if a is taken successively as 5 (black body), 6, 7, and 8, the temperatures being so chosen that the relative emission at $\lambda = 0.5\ \mu$ and $\lambda = 0.7\ \mu$ is always the same. Fig. 2 shows corresponding comparative curves for graphite, tantalum, and tungsten as computed from Coblentz' data, and, finally, Fig. 3 shows the actual observed curves for tantalum and tungsten as compared with untreated carbon corresponding to a color-match and plotted to the same ordinate at $\lambda = 0.5\ \mu$. The actual temperatures at which the comparisons in the three figures were made are not exactly the same, but the differences would produce only a negligible effect on the conclusions deduced from a comparison of the three figures.

[1] Coblentz, *Bulletin of the Bureau of Standards*, **7**, 197, 1912.

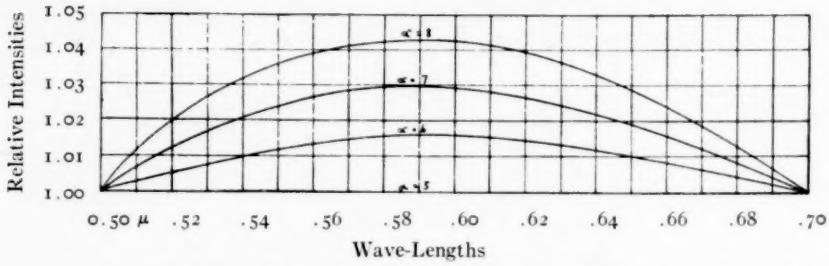

FIG. 1.—Relative distributions of energy in the visible spectrum for various values of a in the generalized Planck equation for temperatures, such that $J_{.5\mu} \div J_{.7\mu}$ is constant.

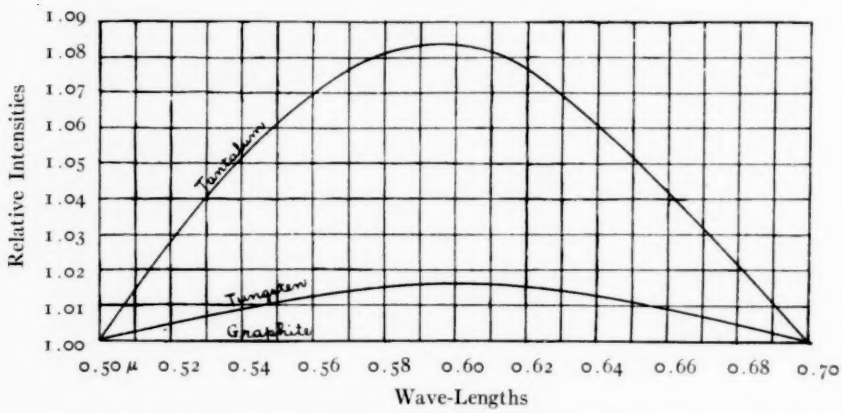

FIG. 2.—Relative distributions of energy in the visible spectrum for graphite, tantalum and tungsten, computed from reflection coefficients, and for temperatures such that $J_{.5\mu} \div J_{.7\mu}$ is constant.

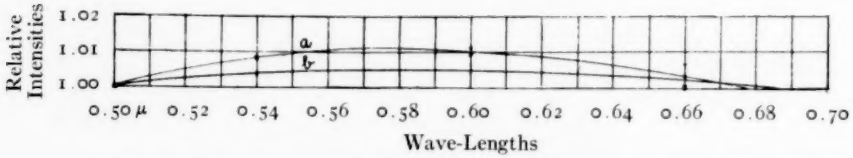

FIG. 3.—Relative observed distributions of energy in the visible spectrum for tantalum and tungsten as compared with untreated carbon for temperatures such that $J_{.5\mu} \div J_{.7\mu}$ is approximately constant.

a = Tantalum { · Observations by F. E. C.; o Observations by A. G. W.

b = Tungsten { × Observations by F. E. C.; + Observations by A. G. W.

In Fig. 3 the observations of two different observers are plotted and show clearly the agreement between the integral light color-matches and the corresponding spectrophotometric relations. A comparison of the curves in this figure with those in Fig. 2 shows no inconsistency in the case of carbon or tungsten between the relative emissivities obtained by Coblentz at room temperatures and those indicated by the relative emission curves obtained at glowing temperatures. In the case of tantalum there is a pronounced discrepancy, but this is not unexpected in view of the difficulties encountered by Coblentz in securing a satisfactory reflecting surface of tantalum.

No attempt has been made to find a mathematical expression for the function $F(\lambda)$.

Before passing on to a discussion of the experimental data regarding Criterion II, it is of interest to consider certain of the data before us in connection with previous observations of the author and others[1] on the relative selectivity of various lamp filaments. The data on the reflectivity of carbon, tantalum, and tungsten showing an increasing emissivity with decreasing wave-length from the deep infra-red to the shortest wave-length in the visible where measurements were made, confirm the assumption made in the previous work regarding the relative temperatures at color-match. It was assumed at that time as highly probable that when tantalum or tungsten was at a color-match with a black body the latter was at the higher temperature. This must be true if the emissivities of tantalum and tungsten increase with decreasing wave-length throughout the visible spectrum as indicated in Coblentz' measurements. Moreover, the measured differences in lumens per watt between untreated carbon and the metals at a color-match must represent, as stated in the previous paper, the minimum effect of selectivity on the relative luminous efficiencies of these substances.

The data in the previous paper which showed that the relative selectivity, as indicated by the relative lumens per watt at color-match, decreased with increasing temperature is of interest in connection with the temperature change of the emissivity in the long wave-lengths. If a black body is compared with a selective body

[1] Hyde, *Jour. Frank. Inst.*, **169**, 439, 1910; and *ibid.*, **170**, 26, 1910.

by measuring the relative lumens per watt at color-match at both low and high temperatures, then if the emissivities of the selective body for different wave-lengths are independent of the temperature throughout the entire spectrum, it can readily be shown by analysis that the relative selectivity, as indicated by the relative lumens per watt, would appear to increase with the temperature. The observed differences for tantalum and tungsten as compared with untreated carbon were on the contrary found to decrease by a somewhat larger amount. The explanation of this would seem to depend on the increasing emissivity in the long wave-lengths with increasing temperature.

Criterion II

Criterion II is fulfilled if it is found experimentally that k_λ, the ratio of the relative change in the emission, J, at some wave-length, λ, to the relative change in the total emission, E, is inversely proportional to some root of the total energy, E (see equations 33 and 34)

$$\frac{(k_\lambda)_1}{(k_\lambda)_2}=\left(\frac{E_2}{E_1}\right)^{1/\beta}.$$

The fulfilment of the criterion rests on the constancy of β.

In most of the experiments measurements were made of k', the ratio of the relative change in the integral light or ordinary candle-power, I, to the relative change in the supplied watts, W, rather than of k_λ as defined above. It is then necessary to correct the relative change in candle-power to the relative change in emission in some wave-length, λ, and to correct the relative change in supplied watts to the relative change in total emission. The latter correction is quite small in all ordinary lamps, and was therefore neglected in most of the present measurements as disproportionately small compared with other sources of error.

The other correction, however, needs further consideration. In the case of a black body obeying Planck's law, the following relation holds in the visible spectrum at glowing temperatures (equations 13 and 23):

$$k_\lambda\equiv\frac{dJ_\lambda}{dE}\ \frac{E}{J_\lambda}=\frac{dJ_\lambda}{dT}\ \frac{T}{J_\lambda}\div\frac{dE}{dT}\ \frac{T}{E}=\frac{1}{4}\ \frac{C_2}{\lambda T}. \qquad (42)$$

The relative change in intensity, J_λ, at any wave-length, λ, corresponding to a given change in total emission, E, is inversely proportional to the wave-length. The relative change in total candle-power, I, will be some mean value of k_λ defined by the following equation:

$$k' = \frac{\int_0^\infty k_\lambda J_\lambda S_\lambda d\lambda}{\int_0^\infty J_\lambda S_\lambda d\lambda} \tag{43}$$

in which S_λ is the sensibility of the eye for the radiation of wave-length λ. Knowing the function S_λ, and also the distribution of energy at the given temperature, i.e., J_λ, it is possible to compute the relation between k' and k_λ for any wave-length and at any temperature. At each temperature there is some wave-length λ_0 for which $k_{\lambda_0} = k'$. As the temperature is raised, λ_0 becomes smaller and smaller, i.e., the relative change in white light increases more rapidly than the relative change in any monochromatic light, which latter increases inversely as the temperature.

Assuming J_λ, the distribution of energy in the spectrum of the black body, to be given by the Planck equation with $C_2 = 14{,}500$, and S_λ, the sensibility function, to be represented by the curve in Fig. 4, k' (equation 43) has been computed for different temperatures. The results of this computation are given in Fig. 5, curve *a*.[1] The data, as given in this curve, are not in the most convenient form for the present use. It is seen (equation 42) that

$$k_\lambda T = \tfrac{1}{4}\frac{C_2}{\lambda} = \text{constant}. \tag{44}$$

Hence $k'T$ (equation 43) will vary with the temperature by an amount proportional to the change in the ratio of k' to k_λ at any given wave-length. In curve *b* are plotted the values of $k'T$ against the temperature. The percentage difference between the ordinate of this curve at any temperature T, and the ordinate at some given temperature T_0, will be equal to the percentage correction which must be applied to the white-light coefficient k' at T in order to make

[1] This curve is similar to that computed by Eisler (*E.T.Z.*, **25**, 189 and 443, 1904), but depends on a somewhat different sensibility curve which is probably more nearly correct.

the ratio of the white-light coefficient k' at T to the white-light coefficient k'_0 at T_0 the same as the ratio of the values of any monochromatic coefficient at the same two temperatures.

Curve b is plotted to give the corrections for a black body at different temperatures, but it is equally applicable to any body which has a constant emissivity in the visible spectrum and which has a distribution of energy in the visible spectrum approximately the same as that of a black body at some temperature. For such a

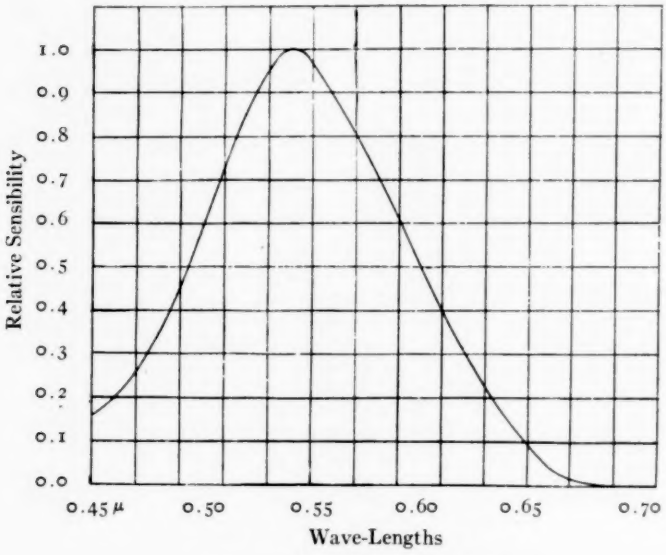

FIG. 4.—Sensibility curve of the eye

radiating body the relative corrections to two k''s at two different temperatures, in order to make the ratio equal to that of the two values of some monochromatic coefficient k_λ, is the same as that for the black body at such temperatures as correspond to color-matches with the radiating body under investigation. In order then to apply curve b to carbon, tantalum, and tungsten it is necessary to know the temperatures of the black body at which it has the same colors as these substances under given conditions of operation.

In earlier work of the author on the radiation of metals certain colors, i.e., certain distributions of energy in the visible spectrum, were arbitrarily fixed by arbitrarily chosen voltages of a given carbon lamp. Thus a certain 110-volt carbon lamp was operated successively at 75, 100, and 125 volts, and an attempt was made to determine the temperatures of a black body giving the same energy distribution in the visible spectrum. The determination at 75 volts was made by direct comparison with a black body, and

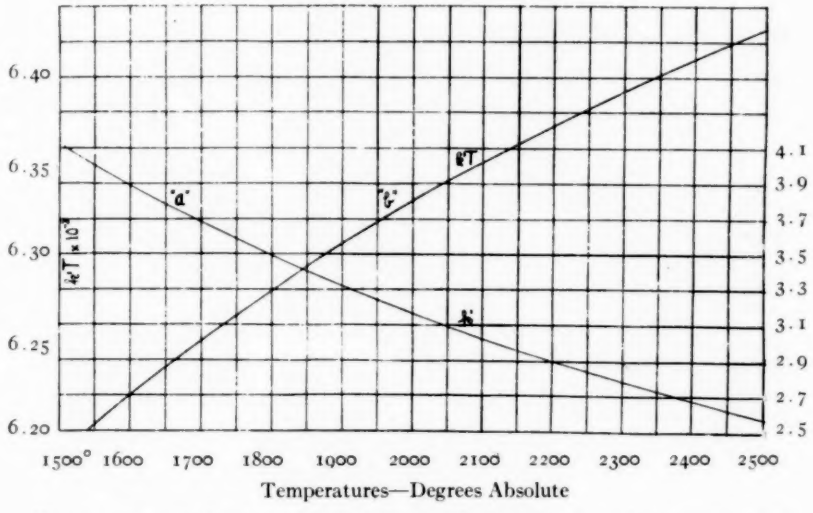

FIG. 5.—Computed values of k' and $k'T$ for a black body at various temperatures

gave as a result 1689° abs. (1416° C.+273°). The determinations at 100 and 125 volts were made by use of a Lummer-Brodhun spectrophotometer and gave the values 1886° abs. and 2058° abs. At the time these measurements were made, however, no investigation had been made of the effect of impurity of spectrum owing to the finite width of the spectrophotometer slits, and so these temperatures, as stated when they were published, are subject to correction. New determinations of these temperatures, using the results of the author's investigation[1] of the effect of the slit-width correction, have led to the following values, assuming the lowest

[1] *Physical Review*, **34**, 233, 1912, and *Astrophysical Journal*, **35**, 237, 1912.

temperature to be 1690° abs., viz., 1935° and 2130° abs. To the original three arbitrarily chosen "colors" corresponding to the colors of a black body at 1690°, 1935°, and 2130° abs. have been added a fourth corresponding to the color of a black body at 2295° abs.

Knowing the watts of the carbon, tantalum, and tungsten lamps corresponding to these colors it is possible, by reference to curve *b*, Fig. 5, to plot the corrections which must be applied to the

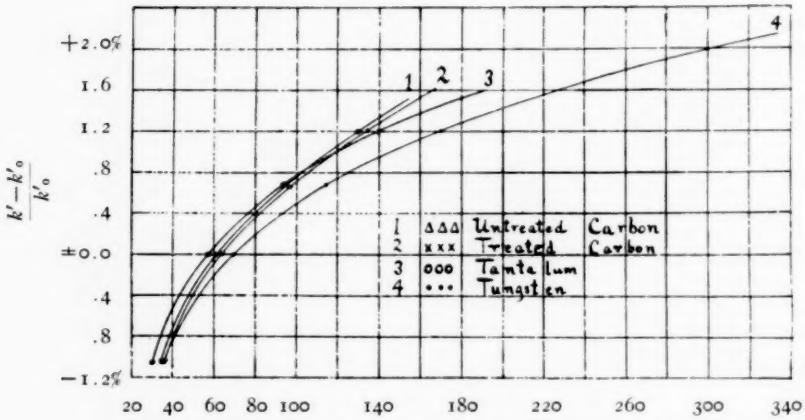

FIG. 6.—Corrections to be applied to observed k' (integral light) in order to get $k'_{\lambda=\lambda_0}$ (monochromatic light, λ_0 being so chosen that $k'=k'_{\lambda_0}$ at color match with a black-body at 1935° absolute).

observed k' at any watts to make it comparable with the k' at some other watts so that the ratio of the corrected white-light coefficients may be the same as that of the monochromatic coefficients, and hence applicable in equation 33:

$$\frac{(k_\lambda)_1}{(k_\lambda)_2}=\left(\frac{E_2}{E_1}\right)^{1/\beta}.$$

In Fig. 6 are plotted such correction curves, the abscissas being in percentage watts, where 100 per cent indicates a specific consumption of 3.5 watts per mean spherical candle for each of the four types of lamps, untreated carbon, flashed carbon, tantalum,

and tungsten. The ordinates give percentage corrections which must be applied to the observed values of k' corresponding to given percentage wattages. Since only relative corrections are of interest, the correction at the color corresponding to a black body at 1935° is taken as zero for each substance. In other words, the monochromatic coefficient k_{λ_0} to which the white-light coefficients are corrected is that which is the same as the white-light coefficient at 1935°. At colors corresponding to higher black-body temperatures the white-light coefficients are relatively too large and must hence be reduced. At colors corresponding to lower black-body temperatures the white-light coefficients must be increased.

TABLE II

DATA FOR UNTREATED CARBON

W (Relative Watts)	Observed k' (White Light)	Corrected k' (Monochromatic Light)	β	$W^{1/\beta}$ (in Terms of Value at 90)	$1/k$ (in Terms of Value at 90)
30	3.593	3.636		.757	.758
35	3.467	3.501	4.44	.789	.788
40	3.352	3.376	3.67	.816	.817
50	3.180	3.190	3.94	.863	.864
60	3.050	3.051	4.10	.903	.904
70	2.950	2.944	4.32	.939	.937
80	2.857	2.846	3.94	.971	.969
90	2.774	2.758	3.75	1.000	1.000
100	2.700	2.680	3.68	1.027	1.029
110	2.627	2.603	3.27		
			Mean 3.99		

These corrections, given in Fig. 6, are applied to the observed white-light coefficients as recorded in Tables II to V. The method used in determining the values of k' is fully described in two papers[1] by F. E. Cady.

Table II contains the average data observed on four untreated carbon lamps. In the first column are given the relative watts expressed in terms of the watts corresponding to 3.5 watts per mean spherical candle taken as unity. In the second column are given the observed values of k', the percentage change in candle-power corresponding to 1 per cent change in watts. In the third column are given the corrected values of k' in accordance with the correction

[1] *Loc. cit.*

curves given in Fig. 6. The fourth column contains the computed values of β for the successive intervals, obtained by application of equation 33 which may be written in the form

$$\beta=\frac{\log E_{n+1}-\log E_n}{\log k'_n-\log k'_{n+1}},$$

or, putting for the total emission, E, its equivalent, W, the supplied watts,

$$\beta=\frac{\log W_{n+1}-\log W_n}{\log k'_n-\log k'_{n+1}}. \tag{45}$$

The weighted mean value of β over the entire range is given at the bottom of this column. The comparatively large variations in β

TABLE III

DATA FOR FLASHED CARBON

W (Relative Watts)	Observed k' (White Light)	Corrected k' (Monochromatic Light)	β	$W^{1/\beta}$ (in Terms of Value at 80)	$1/k'$ (in Terms of Value at 80)
40	3.330	3.356		.845	.840
50	3.162	3.173	3.98	.892	.889
60	3.033	3.035	4.10	.933	.929
70	2.928	2.923	4.10	.968	.966
80	2.831	2.821	3.76	1.000	1.000
90	2.749	2.734	3.76	1.029	1.032
100	2.684	2.665	4.12	1.056	1.059
110	2.627	2.604	4.11	1.081	1.083
120	2.580	2.554	4.49	1.104	1.104
130	2.541	2.512	4.83	1.126	1.123
140	2.506	2.474	4.86	1.147	1.140
			Mean 4.10		

are evident, but when it is considered that over the small intervals employed an error of 0.1 per cent in k' is sufficient to make an error of as much as 5 per cent in β, the magnitude of the differences is surprisingly small. It is evident, however, even with the large variations, that there is no consistent change in β. This is more strikingly shown, as is also the conformity of the data to equation 33, when a comparison is made between $E^{1/\beta}$ and $1/k'$. In the fifth column are given values of $W^{1/\beta}$ expressed in such an arbitrary unit that this quantity has the value of unity corresponding to $W=90$, β being the weighted mean value of the different β's for the various intervals. In the sixth column, for comparison with this

are given the corrected values of $1/k'$ plotted in terms of a corresponding arbitrary unit. If the data conformed to equation 33, which expresses Criterion II, the figures in these two columns would be identical. The agreement is seen to be most satisfactory.

TABLE IV

DATA FOR TANTALUM

W (Relative Watts)	Observed k' (White Light)	Corrected k' (Monochromatic Light)	β	$W^{1/\beta}$ (in Terms of Value at 100)	$1/k'$ (in Terms of Value at 100)
50	2.793	2.798		.863	.862
60	2.701	2.696	4.91	.897	.895
70	2.618	2.610	4.75	.927	.924
80	2.541	2.529	4.23	.954	.954
100	2.430	2.412	4.70	1.000	1.000
130	2.302	2.276	4.54	1.058	1.060
160	2.205	2.174	4.53	1.105	1.110
190	2.141	2.107	5.49	1.146	1.145
			Mean 4.71		

TABLE V

DATA FOR TUNGSTEN

W (Relative Watts)	Observed k' (White Light)	Corrected k' (Monochromatic Light)	β	$W^{1/\beta}$ (in Terms of Value at 100)	$1/k'$ (in Terms of Value at 100)
37.64	2.85	2.876		.849	.850
53.76	2.71	2.719	6.35	.901	.899
80	2.544	2.539	5.80	.963	.962
90	2.496	2.487	5.69	.982	.983
100	2.456	2.444	6.04	1.000	1.000
110	2.420	2.405	5.93	1.016	1.016
120	2.389	2.371	6.10	1.031	1.031
140	2.329	2.307	5.63	1.058	1.059
170	2.260	2.233	5.96	1.093	1.095
200	2.203	2.172	5.86	1.123	1.125
230	2.158	2.123	6.14	1.150	1.152
260	2.119	2.081	6.14	1.174	1.174
300	2.070	2.029	5.65	1.203	1.204
			Mean 5.95		

Nowhere is there a difference of more than 0.2 per cent, which is within the observational error.

Similar data for flashed carbon, tantalum, and tungsten are given in Tables III, IV, and V. The agreement for these substances is almost as good, though the values of β are seen to be quite different. That the four filament materials quite closely fulfil Criterion II is indicated strongly by the data presented here.

In Table VI are collected the observed average values of β for the different substances given to the nearest figure in the first decimal place. It is significant that the value of β for untreated carbon should come out exactly 4.0, identical with that of a black body. For flashed carbon the value is 4.1, for tantalum 4.7, and for tungsten 6.0, the differences being in the direction one would expect from the observations of Coblentz and the earlier work of the author on selectivity. But it cannot be emphasized too strongly that the actual values of β as given are not insisted upon as being highly accurate, particularly in the case of tungsten, which gives evidence of a change in emissivity with temperature, even in the visible spectrum. A very slight uncertainty in the determination of k' would also be productive of error, so that the values of β

TABLE VI

AVERAGE VALUES OF β

Filaments	β
Untreated carbon	4.0
Flashed carbon	4.1
Tantalum	4.7
Tungsten	6.0

actually obtained are surprisingly close to what might have been anticipated from accumulated evidence from other sources.

The present author[1] and others in the preceding work on the selectivity of radiation of filament materials found flashed carbon to be slightly more selective than untreated carbon; tantalum was found to be still more selective, and tungsten quite a little more selective than tantalum. Coblentz' reflection data indicate the same order and approximately the same degree of selectivity and the β's found in the present investigation are not inconsistent with the earlier data, though this would still be true if the value of β for tungsten was 5.5 instead of 6.0, a value more probable from some considerations.

To make sure that the observed value of β for tungsten was not in error owing to errors in observation, the determination was repeated using monochromatic light and making the measurements

[1] *Loc. cit.*

on a spectrophotometer, two observers making independent sets. A special lamp was employed and the end losses were measured so that only the central portion of the filament was in actual use. The two sets of measurements were in excellent agreement and gave as a mean value of β over a fairly large interval of watts the value 6.2. If, therefore, the change in energy radiated does not proceed as rapidly as the sixth power of the temperature, the discrepancy would seem to the author to lie in the unwarranted assumption of the constancy of the emissivity of tungsten in the visible spectrum.

Criterion II, which demands the constancy of β in the relation between k_λ and E (equation 33), is seen to be well fulfilled for untreated carbon, flashed carbon, tantalum, and tungsten, within the limits of experimental accuracy, and over the range of temperature within which the observations were made. If Criterion I, taken to indicate constancy of emissivity in the visible spectrum, were accurately fulfilled, then k_λ (or k' corrected) would be proportional to $1/T$ and so Criterion I in conjunction with Criterion II would lead to the generalized Stefan-Boltzmann law in the form

$$\frac{E_2}{E_1}=\left(\frac{T_2}{T_1}\right)^{\beta} \tag{46}$$

where β for the various substances has the values given in Table VI over the range of temperature employed. The theoretical necessity that β approach "4" will be discussed later, but before undertaking this discussion it is interesting to consider the connection between selective emission and the relation between total emission and temperature; in other words, to consider the influence of selective radiation on the exponent β in equation 46.

In the case of a black body the rate of change of intensity of emission with respect to a change in temperature varies with the wave-length. Corresponding to a given increase in temperature the intensity of emission at every wave-length is increased, but the rate of increase varies approximately inversely with λ, being therefore relatively large for the visible or ultra-violet region of the spectrum, and relatively small for the long infra-red rays. The average change for the total emission is, however, according to the

Stefan-Boltzmann law, as the fourth power of the temperature. Now if the emissivity in the shorter wave-lengths for another body is relatively larger than that in the longer wave-lengths, relatively more energy at any temperature will be radiated in the shorter wave-lengths as compared with that emitted in the longer wave-lengths than would be the case for a black body. Hence for a small increase in temperature the rate of increase of the intensity of emission in the shorter wave-lengths would have greater weight in determining the rate of increase of total emission than that in the longer wave-lengths. And since the coefficient of increase is greater for the shorter wave-lengths therefore the total emission would increase more rapidly than as the fourth power of the temperature, as in the case of a black body.

Assuming the emissivities of tungsten as given by Coblentz to remain constant, β was computed by the method outlined above, and yielded as a result the value 5.25, being thus somewhat lower than the value 6.0 found by the other method, notwithstanding the fact that since the emissivities in the longer wave-lengths probably increase with temperature, the selectivity at glowing temperatures is less pronounced than at room temperatures at which the reflection coefficients were measured. There is, however, one other element entering, still supposing the emissivity in the visible to be independent of the temperature, and that is the effect of the changing emissivities in the long wave-lengths. Since these latter increase with the temperature, one would expect the total emission to increase more rapidly with the temperature than would be computed on the assumption of their constancy. Hence corresponding to any given selectivity at any temperature, β would be larger than "4" for the two reasons: (1) the predominating effect of the larger coefficients in the shorter wave-lengths; and (2) the increase in the emissivities in the longer wave-lengths. No attempt has been made to compute the relative magnitudes of these two factors.

As the emissivities in the infra-red become larger, however, the selectivity decreases and so the first of the two factors must decrease. The effect of the second factor will depend on the rate of change of the emissivity in the long wave-lengths at increasing tempera-

tures. This rate of increase must ultimately approach zero as the emissivity in this region approaches that of a black body. As this condition is approached the effect of the first factor, i.e., of the selectivity, must also become negligibly small, since the selectivity will have diminished until it is negligible. Hence, if, for a given radiating substance, one assumes that the substance approaches a black body in its radiation, and that if it should undergo no change of state but should continue to radiate in the same general way, it would ultimately radiate exactly like a black body, then it is clear that β would approach the value "4" as its limit. Indeed it is difficult to conceive how the exponent β in equation 46 can fail to decrease as the temperature is increased indefinitely, since otherwise the emission at sufficiently high temperatures would become greater than that of the black body. This argument for the decrease in β must not be confused, however, with the similar but less rigorous argument for the decrease in a in the generalized Planck equation (equation 24). The generalized Stefan-Boltzmann law (equation 26) contains two constants, σ and a, but since σ is itself a function of a (equation 27) the effect of a decrease in a with increasing temperature is not immediately determinable; it is necessary to consider the effect both in the exponent and in the coefficient.[1]

Since β must ultimately decrease, the question immediately arises why such a decrease would not manifest itself over the range of temperature used in the present experiments. The only answer is that it may possibly be changing continuously but that the magnitude of the change within the prescribed limits of temperature may be less than the experimental errors. With the exception of tungsten, which owing to the uncertainty regarding its conformity to Criterion I is somewhat unsatisfactory, the largest range of temperature is only of the order of magnitude of 600° or 800° as in the case of tantalum, and since β for tantalum is approximately 4.7 it is only necessary to assume that the decrease in β should be 0.2 or 0.3 every 1000° in order to have it reduced to 4.0 at 5000° or 6000° provided there was no change in state. But a change in β of 0.2 or 0.3 over the entire range of temperature used in the present experiments is well within the range of experimental error,

[1] *Loc. cit.*

and, moreover, there is no reason to believe that the radiating body would become a black body at so low a temperature as 5000° or 6000° abs. Of course this question is interlinked with that of the absolute emissivity at a given temperature, and the necessary changes in β are partly determined by this factor.

FILAMENT TEMPERATURES

Nothing has been said so far regarding the actual temperatures of operation of the various filaments. Moreover, there is no intention of entering into a general discussion of this question in the present paper. However, as certain deductions regarding temperature follow immediately from the preceding theory, it seems advisable to refer to them briefly. If Criterion I and Criterion II are both fulfilled, assuming the former to indicate constant emissivity in the visible spectrum, then (equations 35 and 39)

$$k_\lambda = \frac{dJ_\lambda}{dE} \cdot \frac{E}{J_\lambda} = \frac{dJ_\lambda}{dT} \cdot \frac{T}{J_\lambda} \div \frac{dE}{dT} \quad \frac{T}{E} = \frac{C_2}{\lambda T} \div \beta = \frac{C_2}{\beta \lambda T} \tag{47}$$

and so

$$T = \frac{C_2}{\beta \lambda k_\lambda} . \tag{48}$$

Since $C_2 = 14{,}500$ as for a black body, and the other quantities are all derivable from the data given in the preceding paragraphs, it is possible to compute the temperature, T, corresponding to any measured k' corrected to k_λ for some given λ. This computation for untreated carbon at 4 watts per mean horizontal candle, flashed carbon at 3.1 wpc, tantalum at 2.0 wpc, and tungsten at 1.25 wpc, gives the values in Table VII.

The computed temperatures for all except tungsten would seem to be of the right order of magnitude, but it must be remembered that β enters as a first-power factor, and since this quantity is certainly not to be depended upon to an accuracy higher than 5 per cent, the computed temperatures may be in error to that extent, even though the radiators fulfil the conditions quite accurately. The temperature found for tungsten would seem to be entirely too low, but since there is direct evidence that tungsten does not conform to the conditions imposed in the fulfilment of the two criteria, equation 48 is not applicable.

The temperatures of tantalum and tungsten were also evaluated in two other ways, both of them, however, depending on the assumed melting-point temperatures of the two substances. For these the values found by Forsythe[1] were taken, viz., 3070° abs. for tantalum, and 3250° abs. for tungsten.

TABLE VII

NORMAL WORKING TEMPERATURES COMPUTED FROM EQUATION 59

Lamps	Watts per Mean Horizontal Candle	Absolute Temperature (°C.+273°)
Untreated carbon	4.0	2250
Flashed carbon	3.1	2290
Tantalum	2.0	2350
Tungsten	1.25	1910

According to one of these methods the wattage, W_0, corresponding to normal operation was measured and then the voltage was increased very rapidly until the filament melted and the maximum wattage, W, reached just before the filament melted was read. The β-root of the ratio of $W_0:W$ was taken as the ratio of the temperature T_0, of normal operation to the melting-point T. The results obtained were: 2175° abs. as the temperature of normal operation of tantalum, and 2375° abs. as the temperature of normal operation of tungsten. For each metal the mean of four or five determinations was taken, and the agreement among the individual determinations was of the order of magnitude of 20°.

Confining our attention to tantalum, since, as before, no great significance is attached to the data on tungsten, it is to be noted that the assumption is made that $\beta=4.7$, the mean value found for tantalum at lower temperatures. Since β may decrease with increasing temperature, the mean value of β between the temperature of normal operation and that of melting may be slightly smaller, the effect of this being to reduce the computed temperature of normal operation. Moreover, since the watts at the melting-point are measured for the entire filament, whereas the melting takes place at but one point, the measured watts must be too small

[1] Forsythe, *Astrophysical Journal*, **34**, 353, 1911.

by a greater or less amount, and the effect of this is such as to reduce the computed temperature of operation.

The second method is somewhat similar to the preceding. An image of a piece of the filament of the test lamp is focused in the plane of the filament of a comparison lamp, and the latter is viewed through a telescope with a monochromatic transmission screen in the eyepiece, the whole[1] constituting a modified form of the Holborn-Kurlbaum optical pyrometer. The test lamp (tantalum or tungsten) is set at normal operation, and the wattage of the foreground comparison lamp (an untreated carbon lamp for which $\beta = 4.0$) is determined for a match. The voltage of the test lamp is then increased until the filament melts, and a measurement is made, as near the melting-point as possible, of the watts in the foreground carbon lamp corresponding to a match. From these measurements, and a knowledge of the melting-point of the test filament, it is possible to compute the temperature of normal operation on the basis of the following assumptions: (1) that the test filaments comply with Criterion I; (2) that the comparison carbon filament complies with Criterion I and Criterion II, having $\beta = 4.0$; (3) that the temperature of normal operation of the carbon lamp is known.

The results obtained by this method, which is open to some of the criticisms of the previous method, are: 2100° abs. as the normal temperature of tantalum, and 2320° as the normal temperature of tungsten if the normal temperature of untreated carbon is taken as 2100°; and 2130° for tantalum, and 2350° for tungsten if the normal temperature of untreated carbon is taken as 2200°.

It is beyond the scope of the present paper to enter into a discussion of the above temperature measurements or to compare the results with those of other observers. The computations are given only for the purpose of anticipating a reasonable query as to the temperature relations deducible from the given data.

SUMMARY

The principal purpose of the present investigation has been to construct synthetically, so far as may be possible, the laws governing the radiation from untreated carbon, flashed carbon, tantalum,

[1] For a full description of this method and its applications see Hyde, Cady, and Worthing, *loc. cit.*

and tungsten, starting from certain simple photometric and electrical measurements. Two criteria have been established which conduce to certain more or less definite conclusions regarding the laws of radiation of glowing solids, and these criteria have been applied to the filament materials mentioned above.

Criterion I (which see), if fulfilled, points strongly to the constancy of the emissivity of a substance with change of temperature for the region of the spectrum in which it is tested. If not fulfilled it indicates definitely a change of emissivity with temperature. As applied to the various filament materials it has led to the conclusion that the emissivities in the visible spectrum of untreated carbon, flashed carbon, and tantalum are probably constant within the limits of attainable accuracy over the range of temperature employed, but that the emissivity of tungsten in the visible spectrum probably undergoes change with a change in temperature.

The criterion conduces definitely to the following equation representing the distribution of energy in the spectral region in which it is fulfilled:

$$J=C_1 f(\lambda)\frac{1}{e^{\frac{C_2}{\lambda\theta(T)}}}.$$

If the criterion is taken to indicate constancy of emissivity the equation becomes

$$J=C_1 F(\lambda)\frac{1}{\lambda^5 e^{\frac{C_2}{\lambda T}}}$$

which is the Wien equation for a black body, multiplied by the function $F(\lambda)$.

Criterion II (which see), taken in conjunction with Criterion I, leads to the following relation between the total emission and the temperature:

$$E=S'[\theta(T)]^\beta.$$

If Criterion I is taken to indicate constancy of emissivity the above equation becomes:

$$E=ST^\beta,$$

the generalized form of the Stefan-Boltzmann law for a black body. All four of the investigated filament materials conform to the

requirements of Criterion II and hence for all, except tungsten, the latter equation for total emission probably holds within the limits of temperature employed. On this assumption the following values of β have been computed: 4.0 for untreated carbon; 4.1 for flashed carbon; and 4.7 for tantalum. A similar computation for tungsten, neglecting the indicated lack of constancy of its emissivity, gives 6.0. These computed values of β are not insisted upon to a high degree of accuracy.

In addition to these specific results the paper contains a general discussion of the radiating properties of metals.

In conclusion the author desires to express his gratitude to his colleagues, Mr. F. E. Cady and Dr. A. G. Worthing, for valuable assistance in both the observations and computations.

PHYSICAL LABORATORY
NATIONAL ELECTRIC LAMP ASSOCIATION
CLEVELAND, OHIO
April 22, 1912

ELEMENTS OF THE ECLIPSING VARIABLES *W DELPHINI*, *W URSAE MAJORIS*, AND *W CRUCIS*

By HENRY NORRIS RUSSELL

As an illustration of the methods developed in the preceding papers,[1] details are here given of the computation of the elements of three eclipsing variables. Reference will be made throughout to the notation and tables of these papers, to which the reader is referred for explanation.

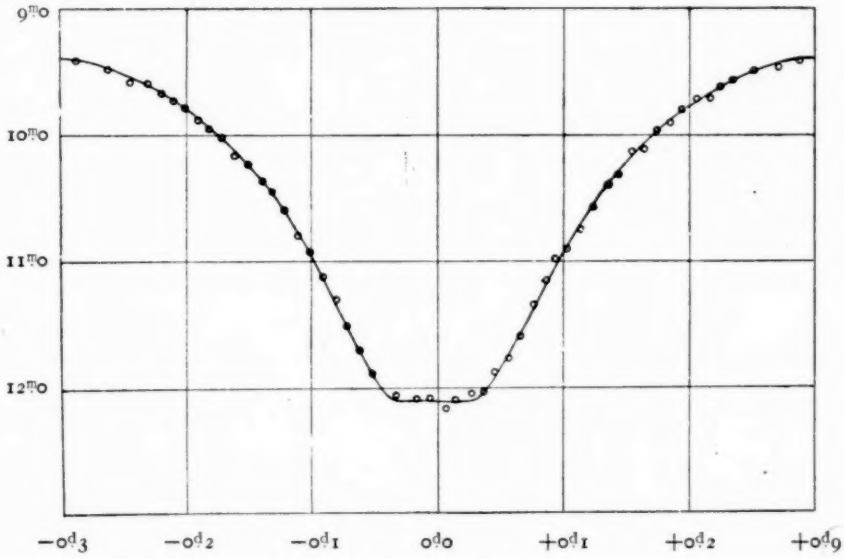

FIG. 1.—Light-Curve of *W Delphini*

I. *W Delphini* ($20^h33^m.1$, $+17°56'$ [1900]) is a good example of the typical *Algol* variable, with a deep primary minimum, showing a constant phase, and little or no secondary minimum. Its light-curve is very well defined by the 500 observations by Professor Wendell, with a polarizing photometer, which are published in the

[1] *Astrophysical Journal*, **35**, 315, 1912, and **36**, 54, 1912.

Harvard *Annals*, **69**, Part 1. These have been combined into normal places as follows, the assumed period of 4.8061 days requiring no alteration:

TABLE I

OBSERVED MAGNITUDES

Phase	Mag.	No. Obs.	O.−C.	Phase	Mag.	No. Obs.	O.−C.
−0d2894	9.41	6	+0m01	+0d0560	11.76	7	+0m01
.2637	9.49	5	+ .02	.0659	11.58	8	+ .01
.2458	9.58	5	+ .04	.0753	11.33	7	− .04
.2306	9.59	4	− .01	.0859	11.14	5	− .02
.2200	9.67	5	.00	.0937	10.97	5	− .05
.2106	9.73	8	+ .01	.1036	10.88	8	+ .02
.2007	9.79	10	.00	.1147	10.73	8	+ .05
.1911	9.88	12	+ .02	.1246	10.56	12	+ .03
.1817	9.95	10	+ .01	.1351	10.39	14	.00
.1718	10.02	8	.00	.1445	10.31	11	+ .04
.1615	10.16	17	+ .04	.1546	10.13	10	− .02
.1506	10.23	14	.00	.1641	10.10	11	+ .04
.1396	10.37	14	+ .01	.1744	9.97	10	.00
.1311	10.44	16	− .03	.1847	9.90	9	+ .02
.1212	10.59	17	− .03	.1941	9.79	9	− .02
.1121	10.78	14	+ .01	.2050	9.71	8	− .02
.1013	10.91	17	− .04	.2157	9.71	6	+ .04
.0906	11.12	14	− .01	.2242	9.63	8	+ .01
.0809	11.30	10	− .02	.2345	9.57	7	.00
.0715	11.51	12	.00	.2507	9.50	7	.00
.0617	11.69	10	.00	.2708	9.48	7	+ .03
.0509	11.88	7	.00	.2811	9.43	4	+ .02
.0313	12.05	5	− .04	.94	9.42	5	+ .02
.0169	12.08	4	− .02	1.90	9.35	5	− .05
− .0082	12.07	7	− .03	2.04	9.41	7	+ .01
+ .0060	12.16	5	+ .06	2.67	9.38	5	− .02
.0139	12.09	4	− .01	3.04	9.42	3	+ .02
.0261	12.03	5	− .07	4.04	9.44	6	+ .04
.0356	12.02	6	− .03	4.48	9.36	7	− .04
+ .0460	11.87	6	− .04				

From the 38 observations outside minimum we find the magnitude during constant light to be $9^m.395 \pm 0.009$. There is no evidence of any change during this period. With a circular orbit, the secondary minimum should occur at phase $2^d.40$. As none of the observations fall within $0^d.27$ of this, they give us no information whether such a minimum exists. The light-curve of the principal minimum is very well determined. The eclipse lasts from about $-0^d.28$ to $+0^d.28$, and there is a short constant period at the middle, of apparently a little less than one-tenth the total duration of the eclipse. The mean of the 20 observations lying within $0^d.02$ of the

middle of eclipse gives for the magnitude at this phase $12^{m}.10 \pm 0.014$. The range of variation is therefore $2^{m}.70$, and the light-intensity at minimum 0.0832 times that at maximum. This shows at once that the eclipse is total, for if it was annular, the companion (even if perfectly dark) must cut off at least 0.917 of the light of the primary, and hence its radius cannot be less than 0.956 times that of the latter. In such a system the duration of the annular phase could not exceed $\frac{0.044}{1.956}$, or 0.022 of the whole duration of eclipse. The observed constant phase is almost five times as long as this.

The brighter star therefore gives 0.9168 of the whole light of the system, and if isolated would appear of magnitude 9.49; while the fainter but larger star which eclipses it gives out only one-eleventh as much light, and when seen alone at minimum is of magnitude 12.10.

We have now to determine the values of $\sin^2 \theta$ corresponding to given values of a_1 (the percentage of obscuration). The loss of light $(1-l)$ will in any given case be $0.9168\, a_1$. The corresponding change in magnitude can be taken at once from Table A. We now draw a free-hand light-curve to represent the observed points, and read from this the epochs t_1 and t_2 at which the magnitude just computed is reached. Taking half the difference of these as the interval t from the middle of eclipse to the given phase, we have $\theta = \frac{2\pi t}{P} = 1.3065\, t$, where θ is expressed in radians, and t in days. Taking the values of $\theta - \sin\theta$ from Table B, we find $\sin\theta$ and $\sin^2\theta$. The results are given in Table II.

From the values of t_1 and t_2 it appears that the observed curve is remarkably symmetrical, and that the actual epoch of mid-eclipse is $0^{d}.0015$ earlier than that assumed by Wendell. The times of beginning and end of the light-changes cannot be read accurately from the curve, and are marked with colons to denote uncertainty.

We have now to find k from the values of $\sin^2\theta$, with the aid of Table II. If A is the value of this quantity when $a_1 = 0.6$, and $A-B$ its value when $a_1 = 0.9$, we have in general $\psi(k, a_1) = \frac{\sin^2\theta - A}{B}$. For every value of a_1, we may now find a value of k, by

inverse interpolation in Table II (or better graphically, by plotting $\psi(k, a_1)$ against k for the tabular values of a_1). When the tabular differences of ψ are small, the resulting values of k are less accurate, and are marked with colons.

As a first approximation we take $A=0.0369$, $B=0.0258$ (which values are derived as above from the observed values of $\sin^2\theta$ corresponding to $a_1=0.6$ and $a_1=0.9$, and partake of any errors which may affect the latter). The resulting values of ψ and k are

TABLE II

a	$1-l$	Mag.	t_1	t_2	θ	$\sin^2\theta$	$\sin^2\theta-A$	$\psi\ (k, a_1)$	k
0.0	0.0000	9^m400	-0^d304:	$+0^d300$:	0.394:	0.1474:	0.1105:	+4.28:	0.56:
.1	.0917	9.505	.2540	.2515	.3304	.1050	.0681	2.64	.504
.2	.1834	9.620	.2285	.2258	.2968	.0860	.0491	1.908	.505
.3	.2750	9.749	.2075	.2030	.2681	.0702	.0333	1.290	.480
.4	.3667	9.896	.1884	.1830	.2426	.0576	.0207	0.802	.462
.5	.4584	10.066	.1682	.1644	.2173	.0462	.0193	+0.361	.36:
.6	.5500	10.266	.1486	.1470	.1931	.0369	.0000	.....	....
.7	.6417	10.514	.1270	.1274	.1661	.0272	− .0097	−0.376	.64:
.8	.7334	10.835	.1070	.1048	.1381	.0190	− .0179	−0.694	.56:
.9	.8250	11.292	.0824	.0788	.1054	.0111	− .0258	−1.000	....
.95	.8709	11.624	.0655	.0624	.0836	.0071	− .0298	−1.155	.58:
.98	.8985	11.884	.0505	.0462	.0632	.0040	− .0329	−1.277	.525
.99	.9076	11.985	.0430	.0390	.0536	.0029	− .0340	−1.318	.528
1.00	.9168	12.100	− .021:	+ .019:	.026:	.0007:	− .0362:	−1.404:	.50:

given in the last columns of Table II. The deduced values of k are fairly accordant, except for those corresponding to values of a_1 near 0.6. A glance at Table II (of the preceding paper) shows that this discrepancy may be almost removed by increasing all the values of ψ by 0.024—which may be done by diminishing A by 0.024 B. Our new value of A is therefore 0.0363. The resulting values of ψ and k are those given under the heading "2d Approximation" in Table III. The values of a_1 near 0.9 now give discordant values of k; but by diminishing B by $2\frac{1}{2}$ per cent, and hence increasing all the computed values of ψ in the corresponding ratio, we obtain a third approximation of a very satisfactory character. Excluding the uncertain data for the beginning of eclipse and of totality, and giving the values of k corresponding to $a_1=0.2$, 0.3, 0.4, 0.95, 0.98, and 0.99 double weight, we find for the mean value of k:

From the upper part of the light-curve..... 0.530
From the middle part of the light-curve..... 0.525
From the lower part of the light-curve..... 0.528

The general mean gives $k=0.528$. If we were to compute the probable error of this mean, from the accordance of the individual determinations, remembering that three quantities—A, B, and k—have been determined from the observations, we would find $k=0.528\pm0.005$. This process is by no means rigorous, but suffices to show that k is very accurately determined. A and B must be known with still greater precision, for changes of only 0.0006 in either one led to the large differences between the successive approximations.

We may now compute a theoretical light-curve, using the constants resulting from our final approximation: $A=0.0363$, $B=0.0252$, $k=0.528$. The results are given in the last columns of Table III, and explain themselves.

TABLE III

	2D APPROX.		3D APPROX.		FINAL LIGHT-CURVE				
a_1	ψ	k	ψ	k	$\psi(0.528,a_1)$	$B\psi$	$\sin^2\theta$	$\sin\theta$	t
0.0	+4.30:	0.56:	+4.40:	0.58:	+4.100	0.1032	0.1395	0.373	0.292
.1	2.665	.512	2.73	.534	2.713	.0683	.1046	.324	.252
.2	1.932	.517	1.974	.538	1.949	.0491	.0854	.292	.227
.3	1.314	.500	1.344	.527	1.348	.0338	.0701	.265	.205
.4	0.826	.503	0.845	.532	0.843	.0212	.0575	.240	.185
.5	0.385	.47:	0.394	.51:	+0.400	.0101	.0464	.215	.166
.6	+0.024	.....	+0.025	.....	0.000	.0000	.0363	.191	.146
.7	−0.352	.48:	−0.360	.54:	−0.358	− .0090	.0273	.165	.127
.8	−0.670	.40:	−0.685	.51:	−0.689	− .0173	.0190	.138	.106
.9	−0.976	.....	−1.000	.....	−1.000	− .0252	.0111	.105	.081
.95	−1.131	.714	−1.157	.564	−1.162	− .0293	.0070	.084	.064
.98	−1.253	.610	−1.282	.507	−1.276	− .0322	.0041	.064	.049
.99	−1.294	.597	−1.324	.512	−1.318	− .0332	.0031	.056	.043
1.00	−1.38:	0.55:	−1.412:	0.48:	−1.389	− .0350	.0013	.036	.028

Plotting the magnitudes computed in Table I against the epochs $-0^d.0015\pm t$, we obtain the computed light-curve. The residuals (O.−C.) are given in the last column of Table I. Their average value, regardless of sign, is $0^m.020$.

We may now proceed to the determination of the elements, which we might just as well have carried out before computing the

light-curve. From Table II*a* of the previous paper[1] we find for $k=0.528$, $\phi_1(k)=0.382$, $\phi_2(k)=0.507$, whence

$$\cot^2 i=\frac{B}{\phi_2(k)}-A=0.0133, \quad \cot i=0.115, \quad i=83^\circ\ 25'.$$

$$r_1^2 \operatorname{cosec}^2 i=\frac{B}{\phi_1(k)}=0.0660, \quad r_1^2=0.0652, \quad r_1=0.256,$$

and finally

$$r_2=kr_1=0.135.$$

In other words, we have, taking the radius of the orbit as unity,

Radius of larger star	0.256
Radius of smaller star	0.135
Inclination of orbit plane	83°25′
Least apparent distance of centers	0.114
Light of larger star	0.0832
Light of smaller star	0.9168

At the middle of eclipse, the larger star overlaps the other by only 0.007 of the radius of the orbit, or about one-twentieth of the radius of the smaller body, so that the eclipse is very nearly grazing. The smaller star gives off eleven times as much light as the other, and exceeds it forty-fold in surface-brightness.

The loss of light at secondary minimum should be k times the light of the fainter star, or 0.023 of that of the system. The corresponding change in stellar magnitude is 0.027, which could only be detected by refined observations.

Applying our formulae for the density of the components, we find, on the hypothesis of equal masses, 0.118 of the sun's density for the density of the smaller and brighter star, and 0.017 for the density of the larger and fainter one. Since the brighter component of a pair differing in magnitude as much as this is usually decidedly the more massive (at least among the visual binaries which alone are so far available for investigation), it is probable that the actual density of the brighter star is about one-third greater, and that of the fainter star one-third less, than the values just computed for them. It is probable that the bright star is at

[1] *Astrophysical Journal*, **35**, 337, 1912.

least ten times as dense as its duller companion. The spectrum of the former is of Class A. It would be very interesting to know that of the latter; but its extreme faintness, and the short interval—only about $1^h.3$—during which it may be observed without interference by the light of the primary, probably put it beyond the range of direct observation.

II. The variable *W Ursae Majoris* ($9^h\ 36^m.7$, $+56°\ 25'$) enjoyed for some time after its discovery the distinction of possessing the shortest known period. The discoverers, Müller and Kempf,[1] published an excellent light-curve based upon 140 observations with the photometer of the Potsdam Observatory. They found the variation continuous, with a period of $4^h\ 0^m\ 13^s.267$, and much more rapid near minimum than near maximum, and called attention to a curious flattening of the light-curve about 45^m on each side of the maximum. They state that the variation cannot be explained by the rotation of a single ellipsoid, but can be well accounted for by almost central eclipse of two ellipsoids nearly in contact—in which case the orbital period must obviously be 8^h. They do not however publish elements, or any comparison of observation with theory.

Five years later, J. M. Baldwin[2] published the results of 260 observations made with the same instrument, which completely confirm the character of the variation, though indicating a somewhat smaller range. The odd and even minima were found to be sensibly equal in depth, and to follow one another at sensibly equal intervals.

The present discussion is based on the results of both observers. The normal places derived from the observations of each, and those finally adopted, are given in Table IV. The differences between the results of the two observers for the same phase vary considerably. On the average, Müller and Kempf appear to have measured the star $0^m.027$ fainter than Baldwin. After correction for this, the average difference, regardless of sign, between the normal magnitudes derived by the two observers is $0^m.040$. Ten of the individual

[1] *Astrophysical Journal*, **17**, 201, 1903; *Sitzungsberichte der Preussischen Akad.*, 1903, p. 173.

[2] *Monthly Notices*, **69**, 78–90, 1908.

differences (in the sense M. and K.−0.027−Baldwin) are positive, and ten are negative, and, when arranged in order of phase, they show ten changes and ten permanences of sign. There does not therefore appear to be any evidence of real change in the star's variation, or of other systematic differences between the observers. The means of their results, with weights proportional to the average number of observations combined into a normal place, has

TABLE IV

M. and K.		Baldwin		Adopted		Corr. for Ellipticity	Rectified Mag.	O.−C.	O.−C.
Phase	Mag.	Phase	Mag.	Phase	Mag.				
$0^h\ 3^m$	$8^m.55$	$0^h\ 3^m.2$	$8^m.48$	$0^h\ 3^m.1$	$8^m.51$	$-0^m.30$	$8^m.21$	$0^m.00$	$0^m.00$
0 11	8.46	0 11 5	8.42	0 11 3	8.44	−0.29	8.15	− .03	− .02
0 20	8.36	0 21 4	8.39	0 20 9	8.38	−0.27	8.11	+ .02	+ .02
0 27	8.30	0 29 8	8.24	0 28 7	8.26	−0.25	8.01	.00	.00
0 35	8.20	0 38 4	8.19	0 37 1	8.20	−0.22	7.98	+ .04	+ .03
0 44	8.20	0 45 5	8.09	0 44 9	8.14	−0.19	7.95	+ .04	+ .03
0 59	8.11	0 53 5 1 0 7	8.06 8.02	0 57 6	8.06	−0.14	7.92	+ .01	+ .01
1 14	8.07	1 12 1 1 21 6	7.99 8.00	1 16 1	8.01	−0.07	7.94	+ .03	+ .03
1 29	7.98	1 32 0	7.97	1 30 8	7.98	−0.04	7.94	+ .03	+ .03
1 44	7.89	1 42 0 1 53 4	7.91 7.95	1 46 8	7.92	−0.01	7.91	.00	.00
2 5	7.88	2 2 3 2 10 9	7.90 7.89	2 6 2	7.89	0.00	7.89	− .02	− .02
2 29	7.96	2 28 9	7.91	2 28 9	7.93	−0.01	7.92	+ .01	+ .01
2 46	7.99	2 41 2 2 55 4	7.96 7.94	2 47 7	7.96	−0.08	7.88	− .03	− .03
3 9	8.05	3 7 6	8.06	3 8 2	8.06	−0.16	7.90	− .01	− .01
3 22	8.08	3 17 7	8.11	3 19 4	8.10	−0.21	7.89	− .02	− .03
3 28	8.16	3 26 7	8.13	3 27 2	8.14	−0.23	7.91	− .03	− .04
3 35	8.23	3 35 2	8.25	3 35 1	8.24	−0.26	7.98	− .04	− .04
3 42	8.35	3 42 4	8.37	3 42 2	8.36	−0.28	8.08	.00	+ .01
3 47	8.47	3 48 4	8.45	3 47 7	8.46	−0.29	8.17	+ .04	+ .04
3 54	8.57	3 54 2	8.43	3 54 1	8.50	−0.30	8.20	+ .02	+ .02

been taken as definitive. Baldwin's normals have received weight 3, except for the last three, which are formed by rearranging observations originally grouped by him into two normals, and have weight 2. Müller and Kempf's normals have all received weight 2.

The observations appear to indicate an asymmetrical light-curve, with the rise to maximum slower than the fall after it; but a symmetrical curve, with minimum at phase $2^m.2$, represents them

with no residual exceeding $0^m.04$, and an average residual, regardless of sign, of $0^m.015$. In view of the differences between the results of the two observers, this may be regarded as quite satisfactory.

Applying the principles of the eclipse theory to this system, we have first to find the ellipticity of the stars, and to "rectify" the light-curve. Reading off from the empirical curve the magnitude at intervals of fifteen minutes, and calculating the values of L^2 and $\cos^2\theta$, we find derive Table V.

TABLE V

t	θ	Mag.	$1-L^2$	$\cos^2\theta$	$0.420\cos^2\theta$	$m-m_0$	M
0^h00^m	0	8.510	0.669	1.000	0.420	0.296	8.206
0 15	$11\frac{1}{4}$	8.420	.609	0.964	.405	.282	8.192
0 30	$22\frac{1}{2}$	8.224	.439	.854	.358	.240	8.150
0 45	$33\frac{3}{4}$	8.106	.303	.691	.290	.186	8.098
1 00	45	8.038	.210	.500	.210	.128	8.038
1 15	$56\frac{1}{4}$	7.985	.129	.309	.130	.075	7.985
1 30	$67\frac{1}{2}$	7.943	.059	.146	.061	.034	7.944
1 45	$78\frac{3}{4}$	7.920	.018	.036	.015	.008	7.918
2 00	90	7.910	0.000	0.000	0.000	0.000	7.910

The values of $1-L^2$ are taken from Table A, with the argument $2(m-m_0)$ where m_0 is the magnitude at maximum. Plotting $1-L^2$ against $\cos^2\theta$, it is found that the last five points lie very nearly on a straight line, whose equation is $1-L^2=0.420\cos^2\theta$. The values of $1-L^2$ computed from this equation, and the corresponding magnitudes, are given in the last columns of Table V, and represent the observed curve for an hour each side of maximum with extraordinary closeness. The ellipticity of the stars therefore accounts for this part of the variation, and explains the flattening of the curve noticed by Müller and Kempf. The eclipse—marked by the observed magnitude being lower than the computed—extends only a little more than 45 minutes on each side of the minimum. The constant $z=\epsilon^2\sin^2 i$ is now determined as 0.420.

If from each observed magnitude we subtract the corresponding value of $m-m_0$ we obtain the material for a rectified light-curve, from which the influence of ellipticity is practically removed. The values of this correction, and the "rectified" magnitudes, are given

in the seventh and eighth columns of Table IV. A plot of the latter shows a light-curve of the ordinary *Algol* type, the magnitude outside eclipse being 7.912, and the eclipse $0^m.30$ in depth, without constant phase.

The eclipses being partial, we must solve our problem with the aid of Tables III and III*a* of the previous paper.[1] We first find the values of $\sin^2\theta$ corresponding to different values of n—that is, to different fractions of the maximum obscuration—which is done, with the aid of a free-hand symmetrical light-curve, just as

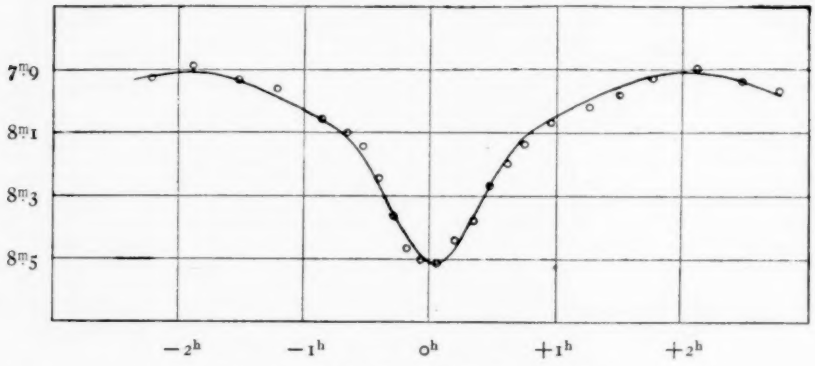

FIG. 2.—Light-Curve of *W Ursae Majoris*

in the case of the preceding star. The loss of light corresponding to the greatest eclipse is 0.240, and we find:

TABLE VI

n	$1-l_1\frac{-}{\pi}$	Mag.	t	θ	$\sin^2\theta$	$\frac{\sin^2\theta}{1-2\cos^2\theta}$	k_1	k_2	t_1	t_2
0.0	0.000	7.912	±47m:	0.616:	0.333:	0.461:	0.435	0.492	±44m1	±48m7
.1	.024	7.938	36 8	.483	.216	.322	.329	.360	37 3	39 5
.2	.048	7.965	31 8	.417	.164	.252	.264	.283	32 6	34 0
.3	.072	7.993	28 4	.372	.132	.208	.218	.228	29 1	29 9
.4	.096	8.022	25 4	.333	.107	.172	.177	.181	25 6	25 9
.5	.120	8.051	22 8	.299	.087	.141	.140	.140	22 7	22 7
.6	.144	8.081	20 0	.262	.067	.110	.107	.104	19 7	19 3
.7	.168	8.112	17 4	.228	.051	.085	.080	.076	16 8	16 3
.8	.192	8.144	13 9	.182	.033	.055	.050	.046	13 2	12 6
.9	.216	8.176	9 7	.127	.016	.028	.025	.022	9 2	8 6
1.0	0.240	8.210	0 0	0.000	0.000	0.000	0.000	0.000	0 0	0 0

[1] *Astrophysical Journal*, **35**, 337–338, 1912.

We have now to determine two constants C and D, so that the equation

$$\frac{\sin^2\theta}{1-z\cos^2\theta}=C\omega_2(n)+D\omega_1(n)$$

shall represent the observed values as closely as possible. The ratio $\frac{C}{D}$ must lie within certain limits, determined by the depth of the minima. From the equations

$$a_0=1-\lambda_1+\frac{1-\lambda_2}{k^2},\quad \frac{C}{D}=\chi(k, a_0, \tfrac{1}{4}),$$

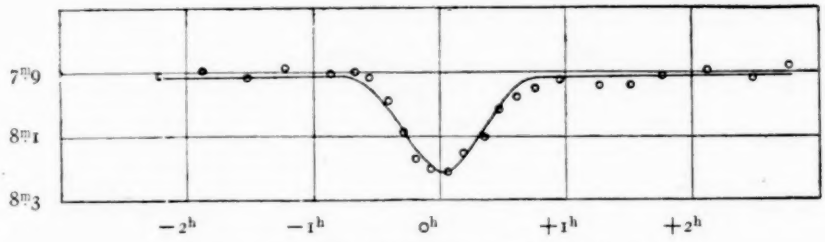

FIG. 3.—Rectified Light-Curve of *W Ursae Majoris*

we find, with the aid of Table III, setting $1-\lambda_1=1-\lambda_2=0.240$, the following values of this ratio, when k and a_0 vary through their whole possible range.

k	1.00	0.90	0.80	0.70	0.65	0.60	0.565
a_0	0.482	0.539	0.618	0.733	0.812	0.911	1.000
$\chi(k, a_0, \frac{1}{4})$	1.817	1.805	1.805	1.802	1.798	1.778	1.720

The ratio $\frac{C}{D}$ must therefore be very nearly 1.80 unless the eclipse is almost total; and, except in this last event, the solution is practically indeterminate.

Setting $C=1.72\ D$, and plotting the values of $\frac{\sin^2\theta}{1-z\cos^2\theta}$ against those of $\omega_1(n)+1.72\omega_2(n)$, we find the best value of D to be 0.140. The corresponding values of $C\omega_2+D\omega_1$ are entered in the column headed k_1 in Table VI. The representation of the observed values is well within the errors of observation. With the

other limiting value, $C=1.82\ D$, we find again that the best value of D is 0.140, and obtain the numbers in the column headed k_2, which appear less satisfactory. If however we derive from k_1 and k_2 the corresponding values of $\sin^2\theta$ and of t (which are given in the last two columns of Table VI) by the equations

$$\sin^2\theta=\frac{k(1-z)}{1-k}, \quad t=\frac{P}{2\pi}\theta,$$

and plot the light-curves corresponding to the two sets of elements, we find that the differences between the magnitudes computed on the two hypotheses are as follows:

Phase	0m	10m	20m	30m	35m	40m	45m
(1)–(2).....	0m000	+0m006	+0m003	−0m005	−0m011	−0m012	−0m005

In the only region where the difference between the two curves exceeds 0m.01, the observed magnitudes during decreasing light lie about 0m.03 above them, and those during increasing light as far below. Normal points formed by combining observations equidistant before and after minimum lie between the two curves. It therefore appears that we cannot hope to distinguish between the two hypotheses (or any of the intermediate ones) on the basis of the existing observations; nor could we do so, even if their already high accuracy were greatly increased. The elements are practically indeterminate within this range—owing mainly to the small magnitude of the partial eclipses.

The residuals, in the sense O.–C., from the light-curves computed on the limiting hypotheses, are given in the last two columns of Table IV. None of the residuals exceeds 0m.04, and their average value, regardless of sign, is 0m.021—exactly the same on the two hypotheses. Since five unknowns[1] have been determined from the 20 observations, the probable error of one normal magnitude is 0m.019. From the average value (0m.040) of the difference between the results of the two observers, the probable error of one of these same normals comes out 0m.018—an excellent agreement. The

[1] The epoch of minimum, magnitude at maximum, and the constants z, C, and D.

eclipse theory therefore furnishes a satisfactory explanation of the observed variation of this star.

We may now compute elements corresponding to the limiting hypotheses, as an indication of the range within which the true values must lie. Our equations are (if θ' is the value of θ at the beginning of eclipse):

$$a_1^2(1-z)[1+kp(k, a_0)]^2=\cos^2 i.$$

$$a(1-z\cos^2\theta')(1+k)^2=\cos^2 i\cos^2\theta'+\sin^2\theta'.$$

$$\epsilon^2=z\operatorname{cosec} i;\quad b_1^2=a_1^2(1-\epsilon^2);\quad a_2=ka_1;\quad b_2=kb_1.$$

The function $p(k, a_0)$ is given in Table I of the preceding paper.

On our two hypotheses we now find:

		I	II
Ratio of radii of stars	k	0.565	1.000
Maximum percentage of eclipse	a_0	1.000	0.482
Function	$p(k, a_0)$	−1.000	−0.161
Value of $\sin^2\theta$ at beginning of eclipse	$\sin^2\theta'$	0.310	0.359
Ellipticity constant	z	0.420	0.420
Major semi-axis of larger star	a_1	0.431	0.366
Minor semi-axis of larger star	b_1	0.326	0.273
Major semi-axis of smaller star	a_2	0.242	0.366
Minor semi-axis of smaller star	b_2	0.184	0.273
Eccentricity of meridian section	ϵ	0.654	0.667
Inclination of orbit plane	i	81°47′	76°30′
Least apparent distance of centers	(cos i)	0.143	0.234
Light of larger star	L_1	0.760	0.500
Light of smaller star	L_2	0.240	0.500
Mean density of the system	(the sun being 1)	2.24	2.22

The mean density of this system is much greater than in the case of any other eclipsing variable, but is not intrinsically improbable. The spectrum of the pair is of Class G; that of most of the eclipsing variables is of Class A, and their densities are in all other cases less than the sun's, and usually very much less. In this case, therefore, the solar type of spectrum appears to go with a late stage of evolution.

If the mass of this system is equal to that of the sun, the relative orbital velocity of the components must be 300 km per second. This star is therefore a very tempting object for spectroscopic investigation, though too faint for any but the greatest instruments. Spectrographic observations would not only give the real dimensions and mass of the system, but would remove the uncertainty

regarding those elements which are derived from the photometric data; for the ratio of the amounts of light emitted by the two stars could be estimated from the relative intensities of their spectra, and this, since the surface brightness of the two is the same, would give the ratio of their radii, (k), and remove all ambiguity.

III. The third variable here considered, *W Crucis* (*C.P.D.* $-58°4151$; $12^h\ 6^m.7$, $-58°\ 14'$), was discovered photographically by Miss Leavitt at Harvard in 1906. The present discussion is based on unpublished observations, which were very kindly communicated to the writer by Miss Leavitt, with the generous approval of Professor Pickering, to both of whom the writer's very hearty thanks are due.

The observations were made by estimation on photographs belonging to the Harvard Photographic Library. Concerning the details of observation, Miss Leavitt says:

> The variable was compared with four stars, and the magnitudes employed were 8.40, 9.00, 9.58, and 10.10. Magnitudes have now been determined by means of comparisons with the South Polar Sequence, on three plates having duplicate exposures. . . . . While the magnitudes of the South Polar Sequence are provisional, it is hoped that they are approximately correct, at least in scale. The magnitudes of the comparison stars for *W. Crucis*, as found by this method, are 8.24, 9.08, 9.68, and 9.99. As the variable, practically, was compared only with the second and third of these stars, between which the earlier and later intervals are 0.58 mag. and 0.60 mag., respectively, and estimates were made only to the nearest tenth of a magnitude, there seems to be no reason to suppose that the range of the variable, as indicated by the light-curve already plotted, differs greatly from the true range. The uniformity of the scale in different parts of the curve evidently depends entirely on the judgment of the observer. An examination of recent photographs gives no conclusive evidence that the period of this variable has changed.

The light-curve is clearly of the *β Lyrae* type; the elements for prediction of the principal minimum being, according to Miss Leavitt, Min. = J.D. $2{,}410{,}158 + 198^d.5\ E$.

The observations as communicated to the writer consisted of 68 normal places, each representing the mean of the estimates on five plates, adjacent in order of phase. In Table VII, to save space, adjacent pairs of these have been combined into single normals, except near principal minimum (between phases -20^d and 20^d),

when the variation is rapid. The normal places in the lower part of the table are therefore of twice the weight of those in the upper part.

From a free-hand curve drawn to represent these normals, it appears that the middle of the principal minimum occurs at about

TABLE VII

Phase	Mag.	Corr.	Rectified Mag.	O.−C.	Phase	Mag.	Corr.	Rectified Mag.	O.−C.
−17.7	9.05	−0.13	8.92	−0.02	55.5	8.90	+0.02	8.92	+0.02
−15.5	9.11	−0.14	8.97	−0.03	66.2	8.92	−0.02	8.90	0.00
−12.5	9.27	−0.15	9.12	+0.02	69.7	8.96	−0.04	8.92	+0.02
−10.1	9.33	−0.16	9.17	−0.02	74.9	9.04	−0.07	8.97	+0.05
− 7.0	9.50	−0.17	9.33	+0.01	80.6	9.03	−0.10	8.93	−0.04
− 4.5	9.45	−0.17	9.28	−0.05	86.5	9.13	−0.13	9.00	−0.01
− 2.5	9.45	−0.17	9.28	−0.05	94.8	9.19	−0.16	9.03	+0.02
− 1.1	9.47	−0.17	9.30	−0.03	100.4	9.16	−0.17	8.99	−0.02
− 0.5	9.55	−0.17	9.38	+0.05	106.3	9.15	−0.16	8.99	+0.03
+ 1.0	9.52	−0.17	9.35	+0.02	118.2	9.00	−0.12	8.88	−0.02
+ 6.5	9.52	−0.16	9.36	+0.03	124.8	8.96	−0.09	8.87	−0.03
+11.4	9.40	−0.13	9.27	+0.02	128.2	8.94	−0.07	8.87	−0.03
+13.2	9.28	−0.11	9.17	0.00	133.8	8.98	−0.05	8.93	+0.03
+17.0	8.94	−0.09	8.85	(−0.21)	142.4	8.93	−0.03	8.90	0.00
+23.2	9.01	−0.07	8.94	+0.04	150.2	8.92	−0.02	8.90	0.00
+26.6	8.96	−0.05	8.91	+0.01	153.5	8.92	−0.03	8.89	−0.01
+28.8	8.96	−0.04	8.92	+0.02	156.1	8.94	−0.03	8.91	+0.01
+33.8	8.90	−0.01	8.89	−0.01	162.5	8.94	−0.05	8.89	−0.01
+38.5	8.90	+0.01	8.91	+0.01	168.0	8.97	−0.07	8.90	0.00
+43.8	8.87	+0.02	8.89	−0.01	176.1	9.02	−0.11	8.91	+0.01
+49.2	8.85	+0.03	8.88	−0.02					

phase 1d, and that of the secondary about 97d. The interval from primary to secondary is shorter than that from secondary back to primary; hence the orbit is eccentric, and periastron passage occurs during the first interval. The maximum light during this interval is 8.87, while near apastron it is unquestionably less—about 8.92. This difference may arise from a combination of causes, connected with the eccentricity of the orbit, such as increased ellipticity of the stars when nearest one another, mutual heating, or tidal effects. It is small enough to justify summary treatment; and therefore the observed magnitudes have simply been corrected for this effect by subtracting the quantity $0^m.025 \cos \theta$. This would in any case

be the first term in the Fourier series by which a more complicated correction might be expressed, and is ample for the present purpose.

Having thus eliminated what may be called the "periastron effect," we proceed, just as in the case of the last star, to find the ellipticity-constant z—assuming the maxima to come halfway between the minima, and taking means of the observed values of L^2 at equal intervals before and after both maxima. Plotting these against $\cos^2 \theta$, we find $z = 0.27$. (In this computation the angle θ,

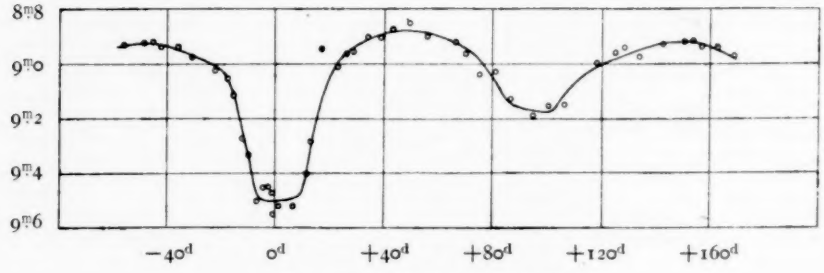

FIG. 4.—Light-Curve of *W Crucis*

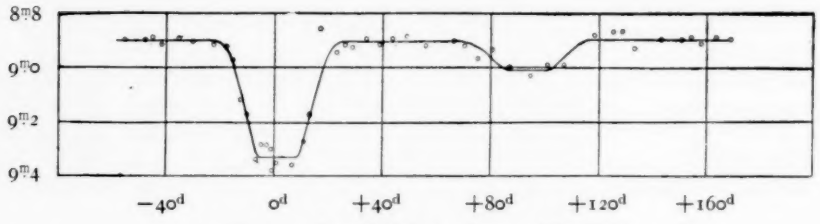

FIG. 5.—Rectified Light-Curve of *W Crucis*

which here expresses the rotation of the stars, is supposed to increase uniformly with the time, in spite of the eccentricity of the orbit.) We may now compute the correction for ellipticity of figure, and adding it to that for the periastron effect, obtain the corrections given in the third column of Table VII, and the rectified magnitudes given in the fourth column.

A plot of these shows a light-curve of the normal *Algol* type, with a conspicuous constant phase during the principal minimum, the rectified magnitude being 9.33 for about fourteen days. The secondary minimum is about $0^m.10$ in depth. The mean rectified

magnitude between the principal minimum and the secondary is 8.902, and between secondary and primary is 8.895—an excellent agreement. The middle of the principal minimum can be fixed with considerable accuracy at phase $1^d.3$. That of the secondary minimum is much more uncertain, owing to the small amplitude of the variation, especially as there are very few observations on the ascending branch of the curve. A curve obtained by reducing the depth of the principal minimum (measured of course in light-intensity, not in magnitude) so that the change in magnitude is 0.11, and with the middle of eclipse at 93^d, gives a good representation of the observations.

We may now determine the orbital eccentricity. The equation

$$e \cos \omega = \frac{\pi}{2P}(t_2 - t_1 - \tfrac{1}{2}P)$$

gives in this case $e \cos \omega = -0.06$. The corresponding quantity, $e \sin \omega$, can be found only by comparing the lengths of the two minima. The secondary is so shallow that it is impossible to determine its duration with any accuracy. The assumption that it is of the same length as the primary represents the observations as well as there is reason to expect. We may therefore set $e \sin \omega = 0$, with relatively high uncertainty. This gives, for the elements of the orbit of the star which is eclipsed at principal minimum, about the other as a center, $e = 0.06$, $\omega = 180°$. Periastron passage occurs halfway between the principal and the secondary minima.

From the discussion in Part III of the preceding paper, it appears that when $e \sin \omega = 0$ the only sensible effect of the orbital eccentricity upon the form of the light-curve for either minimum is to make them very slightly asymmetrical. The minima will be of equal length, and the radii of the stars and inclination of the orbit can be determined from the light-curve of either one, just as if the orbit was circular. We shall see later that the theoretically predicted asymmetry is also practically insensible.

Proceeding to determine the remaining elements, we note that the ratio k of the radii of the components may be found at once from the depths of the minima. The loss of light at the principal minimum is 0.324 of that of the whole system, and that at the

secondary is 0.096. If the principal minimum is assumed to be total, the smaller star must emit 0.324 of the light of the system, and obscure $\frac{0.096}{0.676}$ or 0.142 of the light of the larger star when it eclipses it annularly at secondary minimum; whence $k = \sqrt{0.142} = 0.38$. In the same way we find that if the secondary minimum is total, the value of k is 0.60. To distinguish between these values we must make a solution for the principal minimum, along the same lines as in the case of the first star discussed in this paper. Our fundamental equation is

$$\sin^2\theta - A = B(1 - z\cos^2\theta)\psi(k, a_1)$$

and the numerical data are given in Table VIII.

TABLE VIII

a_1	Mag.	t	$\sin^2\theta$	$\frac{\sin^2\theta - A}{1 - z\cos^2\theta}$	ψ	k	Computed for k 0.38			
							ψ	$\sin^2\theta$	t	m¹
0.0	8.900	23^d1:	0.450:	0.345:	3.92:	0.49:	3.42	0.422	22^d3	8.90
.2	8.973	17.4	0.275	0.148	1.68	0.36	1.70	0.280	17.7	8.92
.4	9.051	15.0	0.210	0.068	0.77	0.40	0.76	0.212	15.1	8.94
.6	9.135	12.8	0.156	0.000	0.00	....	0.00	0.156	12.8	8.96
.8	9.226	10.7	0.110	−0.060	−0.68	0.46	−0.67	0.109	10.7	8.99
.9	9.276	9.6	0.090	−0.088	−1.00	....	−1.00	0.087	9.4	9.00
1.0	9.335	7.8:	0.059:	−0.130:	−1.48:	0.30:	−1.45	0.056	7.6	9.01

From the data for $a_1 = 0.6$ and $a_1 = 0.9$ we find $A = 0.156$ and $B = 0.088$, and thence the values of ψ and k in the sixth and seventh columns of the table. It is evident that the smaller of the two values of k derived above is the correct one, and therefore that the principal eclipse is total.

We may now improve our determination of the other constants. Plotting $\frac{\sin^2\theta - A}{1 - z\cos^2\theta}$ against the new values of ψ, we find the most satisfactory value of B to be 0.092, while A retains its former value. We may then compute the values of $\sin^2\theta$ and t given in the latter part of the table—which are in excellent agreement with those derived from observation. For the secondary minimum, we may use the same values of t, with the magnitudes given in the last column of the table.

The light-curve may now be drawn, and the residuals found for the individual observations. These are given in the last column of Table VII. With the exception of the observation at phase 17.0 (which is evidently affected by some error, and has been neglected in the discussion), the representation is very good. The average value of the residuals, regardless of sign, is $0^m.027$ for the 13 normals of unit weight, and $0^m.018$ for the 27 of double weight. Since nine unknowns in all have been determined from the observations, the probable error of a normal place of unit weight is $\pm 0^m.025$. When it is considered that these normals are means of but five estimates, made only to $0^m.1$, on photographs taken with no special reference to this star, or to photometric use, this accuracy is amazing. The value of the Harvard photographs in the study of all classes of regular variables will evidently be exceedingly great.

The distribution of the residuals, in sign and magnitude, shows no anomalies, and there can be no doubt that the eclipse theory, as here developed, gives a very satisfactory account of the variation of this star.

We may next proceed to determine the elements, using the equations

$$a_1^2(1+k)^2(1-z\cos^2\theta') = \cos^2 i\cos^2\theta'+\sin^2\theta',$$
$$a_1^2(1-k)^2(1-z\cos^2\theta'') = \cos^2 i\cos^2\theta''+\sin^2\theta'',$$

where θ' and θ'' correspond to the beginning of eclipse and of totality. Introducing the numerical values of our constants, we have

$$1.608a_1^2 = 0.578\cos^2 i+0.422,$$
$$0.286a_1^2 = 0.944\cos^2 i+0.056,$$

whence $a_1^2 = 0.270$, $\cos^2 i = 0.0225$.

We may now calculate the asymmetry of the light-curve. The duration of the light-changes is increased on the side toward periastron, and decreased on the opposite side, by an amount ϵ, which, by equation (36) of the preceding paper, is

$$\epsilon = -e\cos\omega\,(\sec\theta-1)(\operatorname{cosec}^2 i-\cos\theta).$$

At the beginning or end of eclipse $\theta = 40°.8$, and, introducing $e\cos\omega = -0.06$, and changing ϵ from circular measure into time,

we find $\epsilon = 0^d.16$. The light-curve does not begin to slope steeply until θ has diminished to about 32°, and at this time $\epsilon = 0^d.06$. The change of magnitude in this interval is hardly more than $0^m.001$, and hence the light-curve is sensibly symmetrical.

Proceeding finally to determine the remaining elements, we have the following summary:

Major semi-axis of the larger star	a_1	0.520
Minor semi-axis of the larger star	b_1	0.442
Major semi-axis of the smaller star	a_2	0.197
Minor semi-axis of the smaller star	b_2	0.168
Eccentricity of meridian section	ϵ	0.526
Inclination of orbit plane	i	81°21′
Least apparent distance of centers	cos i	0.150
Light of larger star	L_1	0.676
Light of smaller star	L_2	0.324
Eccentricity of orbit	e	0.06
Longitude of periastron	ω	180°
Distance of surfaces at periastron		0.22
Distance of surfaces at apastron		0.74
Period	P	$198^d.5$

The closeness of the components, and the considerable variation in the distance of the nearest points of their surfaces, make the observed brightening of 5 per cent at periastron, compared with apastron, not at all surprising. Both components are of remarkably small density. On the assumption of equal masses, we find for the smaller star a density of 3.1×10^{-5}, and for the larger star 1.7×10^{-6}, the sun's density being unity. That is, their mean densities are approximately $\frac{1}{30}$ and $\frac{1}{580}$ of that of air under the standard conditions of temperature and pressure. If one of the stars is denser than the assigned value, the other must be less dense; and neither one can be more than twice as dense as the assigned value.

These results may appear at first sight almost incredible; but the density computed for the larger star is that which the sun would have if expanded to a radius of about 60,000,000 km, or the size of the orbit of *Mercury*. The nebulae, which show sensible disks, must be many times larger—probably many hundreds of

times—and their mean densities correspondingly less, so that this system is probably far advanced beyond the nebulous condition. It might also be questioned whether bodies of such low density could be opaque; but in traversing 120,000,000 km of gas of the mean density of the larger component, the light of its neighbor would be as much weakened as if it had penetrated 200,000 km of air of standard density. A very much smaller thickness would undoubtedly be practically opaque.

The spectrum of this system was examined by Mrs. Fleming, who made the following notes upon it, dated February 3, 1911:

Plate	Date	Expos.	Spectrum	Remarks
B 37591..	June 6, 1907	60m	G pec.	"Composite, showing many bright lines. May be similar to β *Lyrae*. Near max. br.? Lines fine and clear."
B 38957..	May 20, 1908	60m	G pec.	"Composite. A few bright lines visible. Near min.?"

According to Miss Leavitt's elements, the first of these photographs was taken thirty-two days after the principal minimum, outside eclipse, and the second sixteen days before the principal minimum, when the smaller star was partly concealed behind the larger.

The existence of bright lines in the spectrum of bodies of such low density is not surprising, but the fact that the background is of solar type shows that spectra of this class cannot be confined to stars late in the order of evolution; for this system is clearly in a very early stage.

The opportunity to make certain general remarks concerning this system, even though they may involve a certain speculative element, is too tempting to be passed over.

We may first inquire into what sort of a system this would develop under the influence of tidal evolution. If the mass of the system is M, that of the larger star yM, and the angular velocity of rotation and orbital motion is ω, the orbital moment of momentum is $M\omega y(1-y)$ (neglecting the small orbital eccentricity), while the rotational moments of momentum of the two stars are $c_1 My\omega(a_1^2+b_1^2)$ and $c_2 My\omega(a_2^2+b_2^2)$, where c_1 and c_2 are constants

depending on the internal constitution of the stars. For homogeneous ellipsoids, $c=0.2$; but the actual stars are probably strongly condensed toward their centers, so that we will assume $c=0.1$.

Introducing the numerical values of a and b for the two components we find for the ratio of the rotational to the orbital moment of momentum $\frac{0.46c_1}{y}+\frac{0.07c_2}{1-y}$. If $c_1=c_2=0.1$, and the stars are equal in mass, this becomes 0.10; and if the larger star has three times the mass of the smaller, it is 0.19. Tidal action tends to transfer all the rotational momentum into orbital, and the parameter of the orbit increases as the square of the orbital momentum. If this process goes to the limit, the final parameter of the orbit will on our first hypothesis have 1.2 times its present value, and on our second 1.4 times. If during the process the orbital eccentricity should increase to 0.5, the final major axis of the orbit would be 15 per cent greater than the parameter. The final period of the system, on the most favorable hypothesis, would be twice the present period, or a little more than a year.

This system therefore approaches, but does not reach, the hypothetical conditions which must have prevailed in the early stages of evolution of visual binary stars, if the latter originated by the fission of a nebulous mass. A system similar to the present, but of $\frac{1}{30}$ the density, might evolve into a binary pair similar to δ *Equulei*.

In the second place, it is easy to show that any plausible values of the mass and surface-brightness of the system demand that its distance shall be exceedingly great. Assuming, for example, that the mass and surface-brightness of the larger component are equal to those of the sun, we find that its mean diameter must be about 85 times that of the sun, and its light-emission some 7000 times as great, while that of the whole system would exceed 10,000 times the sun's light. Its photographic magnitude at maximum is 8.9, and its visual magnitude probably about 8.2. To appear of this brightness, the sun (whose stellar magnitude is about -26.8) would have to be removed to a distance of 150 light-years, corresponding to a parallax of 0″.021. On the above assumptions

(neglecting possible absorption of light in interstellar space) *W Crucis* would be 100 times as far away—at a distance of 15,000 light-years, with a parallax of 0".0002. These assumptions may of course be greatly in error; but even if we assume a mass $\frac{1}{8}$ that of the sun (which is less than has been found for any star for which the necessary data are at all reliable), and a surface-brightness $\frac{1}{10}$ that of the sun (which seems very small, considering the general similarity of their spectral types), we still find a distance of 2500 light-years. By modifying our assumptions in the opposite direction, without going outside the range of observed stellar masses, the computed distance might be raised to 50,000 light-years. In any case, there can be no doubt that this star is enormously remote.

Spectroscopic study of the system would be of extreme interest; but it is so faint, and so far south, that it is practically out of reach with existing instruments.

The light-curves shown in the figures are simply graphical representations of parts of Tables I, IV, and VII. The curves are in all cases those corresponding to the finally adopted elements. The agreement between theory and observation is excellent. The "rectified" light-curves show how a curve of the *β Lyrae* type is transformed into one of the ordinary *Algol* type by the application of the correction for the ellipticity of the stars.

PRINCETON UNIVERSITY OBSERVATORY
April 10, 1912

ON A NEW METHOD OF DETERMINING THE ANGULAR DIAMETERS OF STARS BY MEANS OF ELLIPTICALLY POLARIZED LIGHT

By S. POKROWSKY

The method which I propose for the determination of the angular diameters of stars by means of the elliptical polarization of light differs essentially from that of Fizeau. It is based on the consideration that, between two rays arriving at any two points from any element of the surface of a distant source of light, there will always be a difference of phase depending on the distance between these points. For the various elements of the surface of the star, this difference will have different values, from zero up to a certain limit determined by the magnitude of its angular diameter. If we polarize all the rays which emanate from the star toward two points not very near together, and if we then superimpose them after turning the plane of polarization of one of the bundles of rays through 90°, we shall obtain, in general, elliptically polarized rays, if the difference of path mentioned above be of the same order of magnitude as the wave-length of light. In the case of an infinitely small angular diameter, the polarization will be rectilinear and will occur in the quadrant through which the aforesaid rotation of the plane of polarization has been effected. Before passing to the theory of my method I shall begin by recalling a case of the composition of two rectilinear vibrations of equal amplitude operating in planes mutually perpendicular. An elliptical vibration results, the axes of which make an angle of 45° with the component vibrations independently of the difference of path which may exist between them. Let us suppose that the two following groups of luminous vibrations proceed together along the axis:

$$\left.\begin{aligned} x_1 &= a_1 \sin 2\pi\left(\frac{t}{T}-\frac{z_1}{\lambda}\right); \quad y_1 = a_1 \sin 2\pi\left(\frac{t}{T}-\frac{z_1+\delta_1}{\lambda}\right); \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_i &= a_i \sin 2\pi\left(\frac{t}{T}-\frac{z_i}{\lambda}\right); \quad y_i = a_i \sin 2\pi\left(\frac{t}{T}-\frac{z_i+\delta_i}{\lambda}\right); \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n &= a_n \sin 2\pi\left(\frac{t}{T}-\frac{z_n}{\lambda}\right); \quad y_n = a_n \sin 2\pi\left(\frac{t}{T}-\frac{z_n+\delta_n}{\lambda}\right). \end{aligned}\right\} \quad \text{I}$$

These superposed vibrations will yield a system of elliptically polarized rays with identical axes. The magnitude of the semi-axes is expressed thus:

$$\left.\begin{aligned} a'_1 &= a_1\sqrt{2}\cos\pi\frac{\delta_1}{\lambda}\,; \quad b'_1 = a_1\sqrt{2}\sin\pi\frac{\delta_1}{\lambda}\,; \\ &\cdots\cdots\cdots\cdots\cdots \\ a'_i &= a_i\sqrt{2}\cos\pi\frac{\delta_i}{\lambda}\,; \quad b'_i = a_i\sqrt{2}\sin\pi\frac{\delta_i}{\lambda}\,; \\ &\cdots\cdots\cdots\cdots\cdots \\ a'_n &= a_n\sqrt{2}\cos\pi\frac{\delta_n}{\lambda}\,; \quad b'_n = a_n\sqrt{2}\sin\pi\frac{\delta_n}{\lambda}\,. \end{aligned}\right\} \text{II}$$

This system of elliptical vibrations, having coincident axes, and at the same time being entirely independent, cannot be studied by means of the process ordinarily employed for studying elliptically polarized light, because the difference in path δ_i will be different for the different vibrations. If we receive this system on a doubly refracting prism in such a manner, for example, that the vibrations along the axes a'_i arrive in the plane of its principal section and, as a consequence, the vibrations along the other axes b' arrive in the plane perpendicular to this section, the prism will separate the two groups of vibrations; two bundles of rays will come out rectilinearly polarized and at right angles. The intensity I_a of one bundle will equal the sum of the intensities Ia'_i of the separate rectilinear vibrations having the amplitudes a'_i; further, the intensity of the other I_b will equal the sum of the intensities $I_{b'_i}$, supposing that the intensity of each separate vibration is proportional to the square of its amplitude.

After these preliminary remarks let us consider the theory of the proposed method. Let us assume that the source of the light studied has the form of a sphere. At a distance L (Fig. 1), exceedingly large in comparison to the other distances which we shall have to discuss, a screen is placed with two openings separated by the distance D. To simplify the reasoning, these openings take the place of two mirrors 1 and 2 of the stellar interferometer described below. From a point M of the surface element of the star with the co-ordinates ϕ and θ, which we take as the center of radiation, the luminous vibrations travel to the screen; between

the rays passing through the homologous elements $\sigma_1\,(\xi_1, \eta_1, \zeta_1)$ and $\sigma_2\,(\xi_2, \eta_2, \zeta_2)$ of the surface of these openings, there will be a certain difference of path $\delta = M\sigma_2 - M\sigma_1$, which can be determined in the following manner: Take the center of the star for the origin of the system of rectangular co-ordinates shown in Fig. 1. The

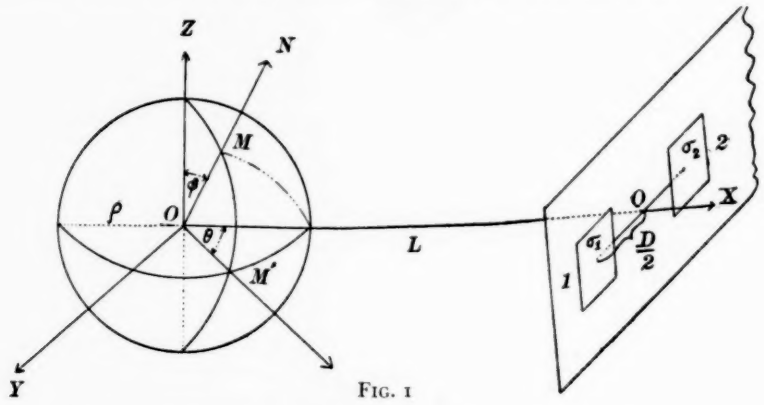

FIG. 1

co-ordinates of the point M and of the homologous elements σ_1, σ_2 of the openings will be

$$\begin{array}{lllll} x & ; & y & ; & z \\ \xi_1 = L\,; & \eta_1 = & \dfrac{D}{2} + \epsilon\,; & \zeta_1 = \zeta_2\,; & \\ \xi_2 = L\,; & \eta_2 = & -\dfrac{D}{2} + \epsilon\,; & & \end{array}$$

where ϵ is the distance along the Y axis of the elements σ_1, σ_2 from the centers of the openings of the screen. We now have approximately

$$\delta = \frac{D}{L} y\,.$$

Introducing the spherical co-ordinates ϕ, θ, where $\phi = \angle ZOM$, $\theta = \angle XOM'$, and confining ourselves to the first powers of the ratio $\frac{2\rho}{L} = \omega$, where 2ρ is the true diameter of the star and ω its angular diameter, we get the definite result

$$\delta = D\frac{\omega}{2} \sin\phi \sin\theta\,.$$

For the different elements of the surface of the star this difference of path will have various values between the limits $-D\frac{\omega}{2}$ and $+D\frac{\omega}{2}$. Moreover, the equality of the amplitudes of the luminous vibrations which have reached the elements σ_1, σ_2 from a given point on the surface of the star may be assumed. The difference of path, with the approximation indicated above, does not depend on the co-ordinates of the elements σ_1, σ_2; for all points of the openings it will be the same. Consequently we may consider the luminous vibrations arriving at these openings as plane waves with a certain difference of path

$$\delta_i = D\frac{\omega}{2} \sin \phi_i \sin \theta_i$$

and of equal amplitude.

In passing the interferometer, these plane waves are polarized, as will be indicated farther on. At the same time the plane of polarization of one ray is turned 90° by means of a half-wave plate; thereupon the waves proceed in the same direction, superposed. Thus for each center of radiation on the surface of the star there will be a plane wave elliptically polarized passing out of the interferometer, the axes of these elliptical vibrations will coincide, and will have the values indicated above (see equation II). This group of plane waves elliptically polarized is entirely analogous to the system (I and II) which we have already examined. It falls on a doubly refracting prism which separates the mutually perpendicular vibrations with the amplitudes a'_i and b'_i. In this manner two systems of plane waves rectilinearly polarized will emerge from the prism, inclined toward each other at a conveniently small angle. The objective of the telescope, upon which the waves fall, will reproduce at its focus two images of the star in question, which will be rectilinearly polarized at right angles. It would be more convenient to place the prism inside the telescope between the objective and its focus, as Rochon did in his telescope.

Now let us find the intensity of each image. Since all the elements on the surface of the star may be considered as independent centers of radiation, the intensity of each image will equal the

sum of intensities of the images of all the separate surface elements. If there are on the surface element n centers of independent radiation—n being in general a very large number—the quantities of luminous energy furnished by the element to the two images, and, consequently, the intensities themselves, will be

$$dI_b = 2 \sin^2 [\beta \sin \phi \sin \theta] k \sum^n a_i^2 ,$$

$$dI_a = 2 \cos^2 [\beta \sin \phi \sin \theta] k \sum^n a_i^2 ;$$

where, for brevity, $\beta = \pi \frac{D}{\lambda} \frac{\omega}{2}$, and k is a certain proportionality factor.

If we designate the radiating power of the surface of the star by i, it is evident that

$$k \sum^n a_i^2 = k' \frac{i \cos (NM\sigma_1)}{L^2} \rho^2 \sin \phi d\phi d\theta$$

where k' is another proportionality factor. The equation

$$\cos (NM\sigma_1) = \sin \phi \cos \theta$$

will be approximately correct. We shall then have

$$dI_b = 2 C \sin^2 [\beta \sin \phi \sin \theta] \cos \theta \sin^2 \phi d\phi d\theta$$

$$dI_a = 2 C \cos^2 [\beta \sin \phi \sin \theta] \cos \theta \sin^2 \phi d\phi d\theta$$

where $C = \frac{ik'\omega^2}{4}$.

Integrating these two expressions we find after some evident transformations

$$I_b = C \left[\pi - \frac{1}{\beta} \int_0^\pi \sin [2\beta \sin \phi] \sin \phi d\phi \right]$$

$$I_a = C \left[\pi + \frac{1}{\beta} \int_0^\pi \sin [2\beta \sin \phi] \sin \phi d\phi \right].$$

The second integral can be represented in the form of Airy's series:

$$\frac{1}{\beta} \int_0^\pi \sin [2\beta \sin \phi] \sin \phi d\phi = \pi S(\beta)$$

$$S(\beta) = 1 - \tfrac{1}{2}\beta^2 + \tfrac{1}{3}\left(\frac{\beta^2}{2!}\right)^2 - \tfrac{1}{4}\left(\frac{\beta^3}{3!}\right)^2 + \tfrac{1}{5}\left(\frac{\beta^4}{4!}\right)^2 - \ldots .$$

The ratio for the intensities of the two images will be

$$i=\frac{I_b}{I_a}=\frac{1-S(\beta)}{1+S(\beta)}.$$

With stars, the value of $\beta=\pi\frac{D}{\lambda}\frac{\omega}{2}$ is usually very small. For example, if $D\sim 10^2$ cm, $\lambda=5\times 10^{-5}$ cm, and $\omega\sim 10^{-7}$ (0".02), we have $\beta\sim 0.3$; consequently one may limit the formula to two members of the series $S(\beta)$. Thus we obtain as an approximation[1]

$$i=\frac{\beta^2}{4}\qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(x)$$

The last expression permits an evaluation in advance of the ratio which will exist between the intensities of the two images of the star, its angular diameter having been given at the outset. We have now seen that if $\omega\sim 10^{-7}$, β will be of the order of 0.3; then

$$i\sim\tfrac{1}{40},$$

that is to say, if an image of the star, for example, appears in the interferometer telescope as a star of the second magnitude, the second image of the same star will be approximately a star of the sixth magnitude. In case the angular diameter is

$$\omega\sim 25\cdot 10^{-9}\ (0.005)$$
$$i\sim\tfrac{1}{640},$$

and the difference of intensity will correspond to a difference of about 7 stellar magnitudes; therefore, if one image in the telescope possesses the intensity of a star of magnitude 0.0, the other will represent approximately a star of magnitude 7.0. This preliminary calculation shows the importance of the intensity of the star under consideration: the more brilliant the star is, the smaller will be the angular diameter that can be detected. Solving the expression (x) for ω, we obtain

$$\omega=\frac{4}{\pi}\frac{\lambda}{D}\sqrt{i}\qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

[1] We shall obtain the same expression if we observe a disk instead of a sphere, because according to Lambert's law the radiation of a sphere is equivalent to the radiation of a disk of the same diameter.

This equation provides the means of calculating the angular diameter of the observed star, if we know D—the distance between the centers of the screen openings—and if, with the aid of an astrophotometer of any kind, we determine i—the ratio of the intensities of the polarized stellar images obtained by the interferometer. Thus we see that the proposed method definitely reduces the determination of angular diameters of stars to a problem of stellar photometry. Its successful application will depend on the lowest feasible value of the ratio i.

But the intensity of the stellar image cannot fall below a certain limit which is determined by the sensitiveness of the eye; we must therefore reckon not only with the ratio of the intensities, i, but also with the minimum intensity, I_o, perceptible to the eye. Suppose it equal to the intensity of a star of magnitude k. Since the intensity I_b of one of the images of the star cannot be less than this limit we put

$$I_b = I_o .$$

We have already seen that with very small angular diameters the two images will show a great difference in intensity; it may therefore be assumed without gross error that the more intense image is formed by the whole luminous energy falling upon the interferometer from the star. Let us assume then that the intensity of the second image will equal the intensity which the star itself had when we examined it with the interferometer telescope. Normally magnified, the intensity of the star increases in the ratio $\frac{S}{\sigma} : 1$, where S is the effective surface of the telescope objective and σ is the area of the pupil. Further, let us assume that of the luminous energy I, when it fell upon the openings of the interferometer, only the part γI reached the eye, γ being a certain fraction usually not very large. It follows that if the star under observation was of the magnitude p to the naked eye, the more intense of the two images, when we observe the star with the interferometer telescope, will appear to be a star of magnitude

$$(p - 2.5 \log \frac{S}{\sigma}\gamma) .$$

The ratio of the intensities i of the images cannot be smaller in the limiting case than that which exists between I_0 and I_p, the intensities of stars of magnitudes k and $(p-2.5 \log \frac{S}{\sigma}\gamma)$. Thus we can give as a minimum value for i

$$i=\frac{I_0}{I_p}=2.5^{-[k-p+2.5 \log \frac{S}{\sigma}\gamma]},$$

and therefore the corresponding minimum value for ω will be

$$\omega=\frac{4}{\pi}\frac{\lambda}{D}2.5^{-\left[\frac{k-p}{2}+1.25 \log \frac{S}{\sigma}\gamma\right]}.$$

The magnitude of the member $1.25 \log \frac{S}{\sigma}\gamma$ depends on the values of σ, S, γ. The last two are determined by the construction and dimensions of the interferometer. This instrument I shall now describe.

The principal part of the apparatus, which determines the optical power by its dimensions, is a rhombohedron of calc spar with polished plane-parallel faces. If two bundles of plane-parallel rays, one of which is polarized in the plane of the principal section and the others in the plane at right angles thereto, fall perpendicularly on the natural face of the spar, the first bundle will traverse it as ordinary rays without refraction, and the second will suffer a parallel displacement as extraordinary rays and will coincide exactly with the first, provided that the size of the beam and thickness of the crystal be properly adjusted to each other. A bundle of elliptically polarized rays can result from the coincidence only when the polarized rays are coherent when they reach the spar: that is to say, in general, capable of interference. It is easy to understand that with natural light one can obtain coherent superimposed ordinary and extraordinary rays in the spar. With such an end in view, it will be more convenient to employ a selenite biplate corresponding to a half-wave. This biplate consists of two plates set side by side (Fig. 2); in one of them the optical axis lies parallel to the line of separation *ab* and in the other it forms an angle of 45°. In the bundle of rays of natural light that have passed through the second half of the biplate, the coherent amplitudes parallel with the

line of separation *ab* are turned through an angle of 90°. This biplate is placed before the spar in such a fashion that the line of separation *ab* is perpendicular to the principal section of the rhombohedron.

After all that has been said, the action of the stellar interferometer which I propose will be clear. Two contiguous bundles of parallel rays of natural light emanating from the star under observation will traverse first (Fig. 3) two plates of glass *V*, *V'* to compensate for the difference in path that will result later between the ordinary and extraordinary rays advancing through the spar with unequal velocities. (The mirrors are supposedly removed from the interferometer.) Having passed through the selenite biplate described above, the rays fall upon the spar, which superposes the ordinary rays that come from one bundle and the extraordinary rays that come from another. The diameter *D* of the bundles will represent the distance between the openings of the screen, which we employed before. If we wish to obtain a greater difference of path, we can direct the rays upon the spar by means of mirrors (I→III; II→IV, Fig. 3), or prisms. In this case D will represent the distance between the centers of the mirrors or the prisms. A slit wide enough to permit the passage of only the elliptically polarized rays is placed before the objective of a medium-sized telescope upon which the rays next fall. They traverse the objective and strike the Wollaston prism introduced before the focus. This prism will yield two images of the observed star; revolving the prism in its plane, we can obtain the position which insures the maximum difference of image intensities.[1] This will serve to show that the more intense image is formed by, one may say, the major axes of

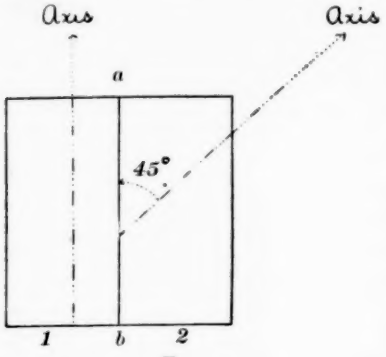

FIG. 2

[1] In this position the principal section of the prism will make an angle of 45° with the principal section of the rhombohedron.

the elliptical vibrations, and the less intense image by the minor. One may find the ratio i of the intensities of these images by means of a Pritchard astrophotometer.

Passing to the evaluation of the sensitiveness of the proposed method, we remark that the transverse section $S=hD$ of the bundles of rays superposed by the spar determines the effective surface of the objective. The size of S will depend on the dimensions of the given rhombohedron. If, to save material, we employ two

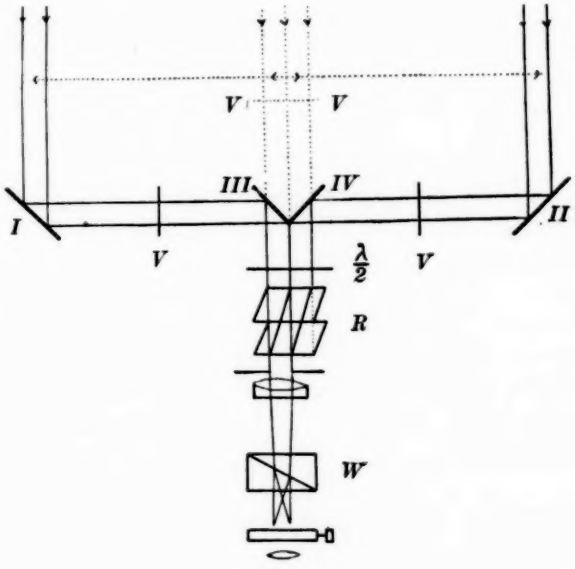

FIG. 3

crystals of the dimensions of about $45\times45\times55$ mm in place of one large one, and place them in contact, S will have the value $30\cdot10=300$ mm^2 ($D=10$ mm; $h=30$ mm). The area σ of the pupil of a normal eye may be supposed equal to 4π mm^2. The coefficient γ, indicating what fraction of the luminous energy falling on the interferometer reaches the eye, is determined in the following manner. Assume that upon the average, on each refracting surface, 5 per cent of the energy is lost by reflection during the normal passage of the rays [for glass, for example, $\left(\frac{\mu-1}{\mu+1}\right)^2=0.04$; $\mu=1.5$]; and

that as a consequence only 95 per cent passes. After the rays have consecutively traversed the plate of glass (2 surf.); the selenite plate (2 surf.); two rhombohedrons (4 surf.); the objective (3 surf.); the double refracting prism (3 surf.); the astrophotometer (3 surf.); and finally the eyepiece (2 surf.)—$(0.95)^{19}$ of the whole quantity of luminous energy received would reach the eye. Assume, further, that the absorption of light in this apparatus amounts to about 5 per cent, we shall find

$$\gamma=(0.95)^{20}.$$

We thus have for the proposed interferometer

$$S=300 \text{ mm}^2; \quad \sigma=4\pi \text{ mm}^2$$

$$2.5 \log \frac{S}{\sigma}\gamma=2.33 ;$$

that is to say, if the star under observation was of a magnitude p to the naked eye, when we observe it in the interferometer telescope the more intense of its two images will appear to be a star of the magnitude

$$p-2.33 .$$

With these data let us determine the minimum value of the angular diameter to be discovered for stars of the first magnitude ($p=1$). We have

$$\omega=0''.46$$

if $k=6$; $D=1$ cm; $\lambda=5\times10^{-5}$ cm.
For the interferometer with its mirrors (allowing for the loss of light by reflection), we shall have

$$\omega=0''.0048$$

if $D=100$ cm.

For stars of the ninth magnitude ($p=9$) we have

$$\omega=0''.19.$$

In the case of the brighter stars, as, for example, *Acturus*, *Vega*, *Canopus*, *Capella*, that is to say, on the average of 0.0 magnitude, this limit will fall still farther to

$$\omega=0''.0031.$$

In the case of *Sirius*

$$\omega = 0''.0017.$$

So far we have supposed that the interferometer itself causes no difference of path between the two rays. If, however, by reason of the lack of symmetry between the two branches, a certain difference of path l is established in the interferometer itself, the ratio i of the intensities will be

$$i = \frac{1-(1-\frac{1}{2}\beta^2)\cos\alpha+\frac{4}{3\pi}\beta\sin\alpha}{1+(1-\frac{1}{2}\beta^2)\cos\alpha-\frac{4}{3\pi}\beta\sin\alpha} \quad \ldots\ldots\ldots\ldots\ldots (2)$$

where

$$\alpha = 2\pi\frac{l}{\lambda}; \quad \beta = \pi\frac{D}{\lambda}\frac{\omega}{2}.$$

If $\alpha = k\pi$, $k = 0, 1, 2, 3 \ldots$ the ratio of the intensities becomes alternately reciprocal. This circumstance allows us to determine the angular diameters more precisely by several consecutive measurements, obtaining a variable difference of phase by inclining the plates in the interferometer; this difference of phase will alternatively invert the ratio of the intensities i. The careful adjustment of the stellar interferometer will require close attention. but the expected results will largely compensate the trouble expended, for they will considerably enlarge the domain of astrophysics which has hitherto depended almost exclusively upon spectrum analysis. By reason of the extraordinary sensitiveness of this apparatus, one may expect atmospheric conditions to affect the stability of the elliptical polarization strongly. For this reason the second image will sometimes become visible even with stars of very small angular diameter. The apparatus can yield positive results only at mountain observatories.

SUPPLEMENT

Perhaps the simplest form of a stellar interferometer would be the following (Fig. 4). A bundle (1) of parallel rays proceeding from a given star traverses the polarizing prism P, similar to the Glan-Thompson prism, which allows only the extraordinary rays E_1 to pass. The second bundle of parallel rays (2), reflected by the

mirror R, traverses the half-wave plate whose principal section is at 45° with the plane of incidence. The rays next enter the aforesaid prism in which the ordinary rays O_2, reflected by the dividing surface ab, coincide with the extraordinary rays E_1. Thanks to the half-wave plate, these coincident rays will be coherent. After their coincidence the rays follow the route already described in our first type of interferometer.

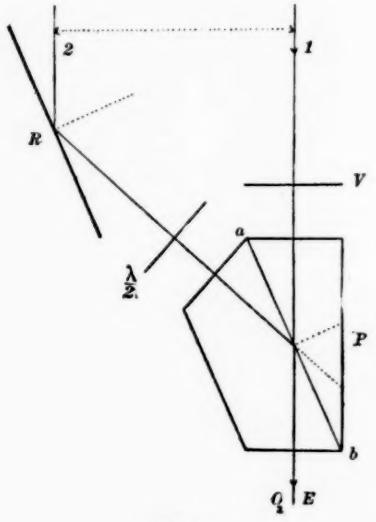

FIG. 4

V is a glass plate compensating the difference of path between O_2 and E_1, introduced by the interferometer itself. The optical power of our second interferometer will considerably surpass that of the first because the effective surface of the modified Glan-Thompson prism noticeably surpasses that of the rhombohedron of calc spar. In this case, one uses the same formulas (1) or (2).

PHYSICAL LABORATORY OF THE ELECTRO-TECHNICAL INSTITUTE
ST. PETERSBURG
March 1910